ASPECTS OF THE GAUGE THEORETICAL STRUCTURE OF GRAVITY AND SUPERGRAVITY

Thesis

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To Ursula, Albert and Astrid.

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ABSTRACT

In this thesis we examine the nature of gravity as a spontaneously broken gauge theory and some of the implications this has on the construction of supergravity theories.

We begin by setting up gravity as a classical gauge theory in an approach which fully exploits the known geometrical requirements and physical constraints on such a theory. We draw heavily on work by previous authors yet present a view which clarifies the situation in many respects.

The second chapter is a review which outlines the deficiency of Einstein's theory of gravitation as a basis for a quantum theory, then introduces (and reviews) supersymmetry as a promising step towards a finite theory of gravitation.

The (minimal) supersymmetric extension of the theory of Chapter I is examined in the third chapter where we consider the same class of invariants as many previous workers. However by careful consideration of the group contraction procedure in the theory we obtain cosmological and gravitino mass terms which may, with suitably chosen numerical coefficients, preserve local supersymmetry and which vanish after group contraction. We also find that models based upon constrained, auxiliary Higgs fields are apparently so restricted that they can have no group contraction limit if they are to contain kinetic terms for the gravitino.

In the final chapter we construct actions for matter multiplets coupled to supergravity, employing the formalism set up in Chapter III. We argue that only the adjoint multiplet may reduce to a flat space-time supersymmetry multiplet and show that it, in fact, coincides with the massless vector multiplet discussed in Chapter II. The coupling of the adjoint multiplet to the gauge fields is examined in detail and compared

with previous approaches. We find a large number of unwanted terms and point out that in the absence of spinor fields the theory reduces to a non-minimal coupling of gravity to electromagnetism. The problems with obtaining a group contraction limit to the theory are demonstrated and we close the chapter with a general discussion on non-linear actions for matter coupled to supergravity.

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INTRODUCTION

The idea that local phase or gauge invariance may determine the form of physical interactions is an appealing one since it provides a comparatively systematic approach to this problem. Yang-Mills gauge theories[(79)] have now attained a status where they have become the basis of the theory of elementary particles, providing the most promising theories to date of the strong and electro-weak interactions. The invariance of these theories is one under a group of transformations in quantum mechanical phase space which are in no way connected with space-time transformations. A geometrical picture of Yang-Mills gauge theories exists in an extended fibre bundle space, where space-time is identified as a cross-section in the bundle, obtained by factoring out the local symmetry group. The geometry of a Yang-Mills bundle, determined by the gauge potentials, does not however reflect on the geometry of space-time itself.

Einstein's theory of gravitation, on the other hand, is a geometrical theory of space-time, where the 'phase transformations' now correspond to physical rotations of vectors. Space-time is a manifold with an affine connection which defines the notion of parallel vectors, infinitesimally separated. A description of affine geometries on a manifold is possible with a special class of fibre bundles, known as fibre bundles with a Cartan connection since, for such bundles, the parallel transport process in the bundle may define parallelism in the cross-section. It is therefore essential that a gauge theory of gravity should have the structure corresponding to a bundle with a Cartan connection.

An important physical constraint which must be imposed on any gauge theory of gravity is to introduce the required metric structure of space-time. The principle of local gauge invariance can only serve to introduce connections and will not reflect, in any way, the metric structure of space-time. In fact, the physics of curved space-time is only understood if it has a metric structure which is locally Minkowski everywhere and covariantly constant with respect to the connection. These conditions introduce non-dynamical constraints into any theory of gravitation which will relate the metric to the connection and require of any field theory that the connection components are non-propagating, auxiliary fields determined by the metric tensor components and any matter fields present.

In the first chapter we develop the above points in a steady, coherent manner to identify the local symmetries of gravity. One clear point to emerge is the existence of a local Lorentz invariance with corresponding gauge potentials which define the parallel transport of spinors and vectors on space-time. These gauge potentials, or spin connection components, are not independent fields however, rather they are determined by the vierbein and matter fields in the theory. The vierbein fields themselves may be regarded as gauge potentials corresponding to the broken generators in a spontaneously broken gauge theory of the anti de Sitter group[69]. We describe a field theoretical model of this broken gauge theory in Appendix D and examine the geometrical structure of the theory in the main text. A comprehensive appendix supplements the first chapter and provides an adequate background for the geometrical arguments presented therein.

Chapter Two serves to motivate our interest in a supersymmetric extension of the gauge theory of gravity presented in the first chapter and to give some background in supersymmetry and supergravity theory in preparation for the later chapters.

In the second half of this thesis we examine various aspects of the spontaneously broken gauge theory of the group OSp(1,4), which contains the anti de Sitter gauge theory of gravity as a sub-theory. Here we simply state that an essential problem with this theory is that the algebraic structure will not allow local supersymmetry to remain unbroken (which is required for a supergravity theory) whilst breaking the de Sitter translational symmetry (which is required for the identification of the corresponding gauge potentials with the vierbein fields). In the gauge theory which we examine the super-gauge generators are 'broken' along with the translations and as a consequence, supersymmetry is realized only through complicated non-linear transformations involving auxiliary Goldstone fields.

CHAPTER 1

GRAVITATION AS A GAUGE THEORY

1) Einstein's Theory of Gravitation

When the theory of "Special Relativity' was proposed by Einstein in 1905[22] it was evident to followers of the theory that Newtonian gravitation with its instantaneous action at a distance would have to be modified. The most important clue for this modification was the known equivalence of the gravitational and inertial mass of any object. In Newtonian physics this equivalence causes no problems with the definition of inertial frames since they may be established at points sufficiently far away from gravitational sources. With special relativity, however, the condition 'sufficiently far away' (spatially) is meaningless since it isn't Lorentz covariant. Einstein realized that the problem must be localized, for a well known example, to the inside of a lift and in such a situation a local free falling, or inertial, frame always exists. The gravitational field thus manifests itself as the inability to set up an extended inertial frame, which on reflection implies that in the presence of a gravitational field space-time is no longer flat.

General Relativity[23] is a geometrical theory of gravitation in which the gravitational field is determined by the 10 component symmetric tensor $g_{\mu\nu}(x)$, the metric tensor of space-time,

$$d\tau^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu$$

where x^μ $(\mu = 0,1,2,3)$ are the coordinates of a point x in space time

$d\tau$ is the physical interval between the two points (events) separated by the coordinate interval dx^μ.

If space-time is flat then extended Lorentz (inertial) frames exist in which,

$$g_{\mu\nu} = \text{diag}\ (1, -1, -1, -1) \equiv \eta_{\mu\nu}\ . \quad \text{(Minkowski metric)}$$

The $g_{\mu\nu}(x)$ play the same role in Einstein's theory as the single scalar potential, $\phi(\underline{r})$ does in Newton's theory:

In Newtonian theory the force $\underline{F}$ on a particle of mass m in a gravitational field $\phi(\underline{r})$ is,

$$\underline{F} = -m\, \underline{\nabla}\, \phi(\underline{r}).$$

Hence the equations of motion of the particle are,

$$\frac{d^2\underline{r}}{dt^2} + \underline{\nabla}\, \phi(\underline{r}) = 0\ .$$

In Einstein's theory the equations of motion of a particle in a 'gravitational field' $g_{\mu\nu}(x)$ are the geodesic equations

$$\underline{\frac{d^2x^\mu}{d\tau^2} + \left\{{\mu \atop \nu\kappa}\right\} \frac{dx^\nu}{d\tau} \frac{dx^\kappa}{d\tau} = 0} \tag{1}$$

where

$$\underline{\left\{{\mu \atop \nu\kappa}\right\} = \frac{1}{2}\, g^{\mu\lambda} (\partial_\nu\, g_{\kappa\lambda} + \partial_\kappa\, g_{\nu\lambda} - \partial_\lambda\, g_{\nu\kappa})} \tag{2}$$

are the Christoffel symbols containing first derivatives in the $g_{\mu\nu}$ and play an analogous role in the equations of motion to the $\underline{\nabla}\, \phi(\underline{r})$ in Newtonian theory.

The solutions $x^\mu(\tau)$ to these geodesic equations, give the particle's space-time coordinates x^μ at 'proper time' τ (measured by a clock co-moving with the particle). The Christoffel symbols also arise as the components of a symmetric affine connection on the space-time manifold[3]. They determine the change in the components of a covariant or contravariant vector under parallel displacement from a point x^μ to an infinitesimally close point $x^\mu + \delta x^\mu$,

$$\delta A^\mu = -\{{}^{\mu}_{\nu\kappa}\} A^\nu \delta x^\kappa \quad \text{and} \quad \delta A_\nu = +\{{}^{\mu}_{\nu\kappa}\} A_\mu \delta x^\kappa .$$

The condition that this parallel transport process is integrable (path independent) is that a given vector A^μ at some point on the manifold will generate, through this parallel transport process, a field of parallel vectors $A^\mu(x)$ which by definition will obey the partial differential equation,

$$\partial_\kappa A^\mu = -\{{}^{\mu}_{\nu\kappa}\} A^\nu .$$

This equation immediately yields the identity, (by differentiating and using $\partial_\lambda \partial_\kappa = \partial_\kappa \partial_\lambda$),

$$\partial_\lambda \{{}^{\mu}_{\nu\kappa}\} - \partial_\kappa \{{}^{\mu}_{\nu\lambda}\} + \{{}^{\sigma}_{\nu\kappa}\}\{{}^{\mu}_{\sigma\lambda}\} - \{{}^{\sigma}_{\nu\lambda}\}\{{}^{\mu}_{\sigma\kappa}\} \equiv 0 . \tag{3}$$

The object on the left hand side is the Riemann curvature tensor, $R^\mu{}_{\nu\lambda\kappa}$, which must vanish if a field of parallel vectors is to have any meaning. Spaces for which $R^\mu{}_{\nu\lambda\kappa} = 0$ are called flat spaces and in such spaces, coordinates x^μ may always be found in which the $g_{\mu\nu}$ become constants[3].

Physically $R^\mu{}_{\nu\lambda\kappa}(x)$ emerges in the equations of geodesic deviation. Consider two particles, free-falling along geodesics $P_1(\tau)$ and $P_2(\tau)$ separated by a small four vector $\xi^\mu(\tau)$.

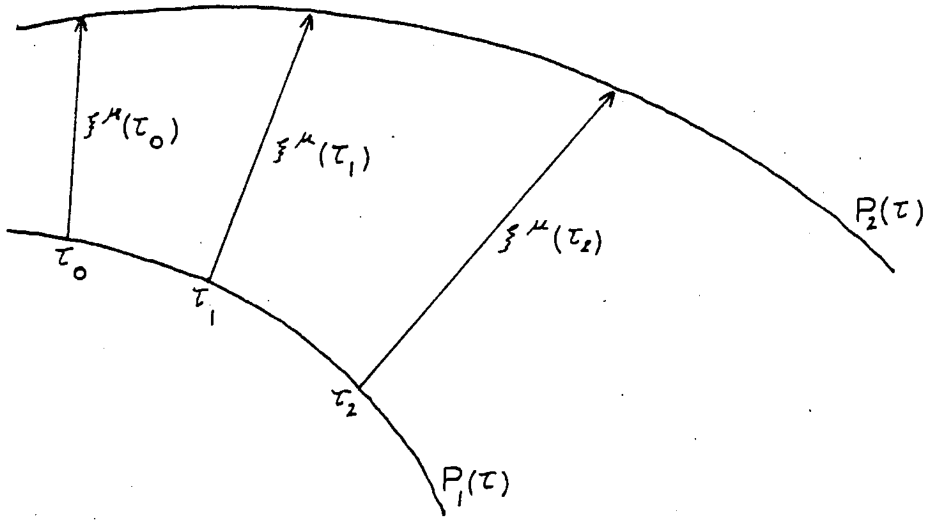


FIGURE 1

The relative acceleration of the two particles is $\dfrac{D^2\xi^{\mu}(\tau)}{d\tau^2}$

where

$$\frac{D\xi^{\mu}(\tau)}{d\tau} = \lim_{\delta\tau\to 0}\left(\frac{\xi^{\mu}(\tau+\delta\tau) - \xi^{\mu}(\text{parallel transported from } x^{\mu}(\tau))}{\delta\tau}\right)$$

$$= \frac{d\xi^{\mu}}{d\tau} + \{{}^{\mu}_{\nu\kappa}\}\,\xi^{\nu}\,\frac{dx^{\kappa}}{d\tau}$$

so that $\dfrac{D\xi^{\mu}}{d\tau}$ is the physical (covariant) rate of change of $\xi^{\mu}(\tau)$ along geodesic $P_1(\tau)$ (with coordinates $x^{\mu}(\tau)$). We then calculate the second derivative and find that,

$$\frac{D^2\,\xi^{\mu}(\tau)}{d\tau^2} = -\,R^{\mu}{}_{\nu\kappa\lambda}(x)\;\xi^{\kappa}(\tau)\frac{dx^{\nu}}{d\tau}(\tau)\;\frac{dx^{\lambda}}{d\tau}(\tau) + O(\xi^2)\ .$$

It is the inhomogeneity of the gravitational field, giving rise to relative accelerations, which physically distinguishes it from inertial fields. We have just seen that it is the Riemann curvature tensor $R^{\mu}{}_{\nu\kappa\lambda}$ which determines relative accelerations so that we identify gravitation as the curvature of space-time. The Christoffel symbols don't distinguish gravitational (geometrical) from inertial (coordinate) effects since they have the inhomogeneous transformation laws[3],

$$\left\{{\mu \atop \nu\kappa}\right\}' = \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial x^{\omega}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\kappa}} \left\{{\lambda \atop \omega\sigma}\right\} + \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial^2 x^{\lambda}}{\partial x'^{\nu}\partial x'^{\kappa}} \tag{4}$$

and so we may always transform to a coordinate system $x^{\mu} \rightarrow x'^{\mu}$ in which the Christoffel symbols vanish at any given point. This is a statement of the equivalence principle that 'locally' gravitational effects may be removed by a choice of reference frame. Just what is meant by locally depends on the accuracy of measurements which an observer may carry out. In principle, the effects of curvature will distinguish gravitational from inertial effects even over infinitesimal distances so that the inertial frames are localized to a single point in space-time.

In summary:

a) <u>If $R^{\mu}{}_{\nu\kappa\lambda} = 0$</u> then there is no gravitational field. Lorentz coordinate frames may be set up in space-time in which $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and so the $\left\{{\mu \atop \nu\kappa}\right\} = 0$. In more general coordinate systems the $\left\{{\mu \atop \nu\kappa}\right\}$ won't vanish but are interpreted as purely inertial fields.

b) <u>If $R^{\mu}{}_{\nu\kappa\lambda} \neq 0$</u> then space-time is curved and global Lorentz frames cannot be established. Frames in which the $\left\{{\mu \atop \nu\kappa}\right\}$ vanish always exist but in the presence of a gravitational field $(R^{\mu}{}_{\nu\kappa\lambda})$ they are localized to points in space-time.

It is interesting to compare these features with their analogues in Yang-Mills theories[79].

Here we recognize the identifications,

$$R^{\mu}{}_{\nu\kappa\lambda} \longleftrightarrow F_{\mu\nu}{}^{i} \quad \text{(Yang-Mills field strength tensor)}$$

$$\left\{ {\mu \atop \nu\kappa} \right\} \longleftrightarrow A_{\mu}{}^{i} \quad \text{(Yang-Mills gauge potentials).}$$

'i' is a Lie algebra index indicating that the fields $F_{\mu\nu}{}^{i}$ and $A_{\mu}{}^{i}$ are the components of the Lie algebra valued fields,

$$F_{\mu\nu} = F_{\mu\nu}{}^{i} X_i \quad \text{and} \quad A_{\mu} = A_{\mu}{}^{i} X_i \quad .$$

The X_i form a basis for the Lie algebra and satisfy the commutation relations,

$$\left[X_i, X_j\right] = f_{ij}{}^{k} X_k$$

The fields A_μ transform under a local gauge transformation, $g(x) \varepsilon$ gauge group, according to

$$\underline{A_{\mu}(x) \xrightarrow{g(x)} A'_{\mu}(x) = g A_{\mu} g^{-1} - g \partial_{\mu} g^{-1}} \quad . \tag{5}$$

The field strengths are obtained from the potentials through the relation

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - \left[A_{\mu}, A_{\nu}\right] \tag{6}$$

so that the F transform homogeneously,

$$\underline{F_{\mu\nu} \xrightarrow{g(x)} g F_{\mu\nu} g^{-1}} \quad . \tag{7}$$

A similar summary of features to that carried out for the gravitational field now reads:

a) If $F_{\mu\nu}{}^{i} = 0$ then there exists a class of gauges in which $A_{\mu} = 0$ everywhere. In a more general gauge $A_{\mu} = g \partial_{\mu} g^{-1}$, yet produce no physical effects (see the Aharonov-Bohm experiment[19]).

b) If $F_{\mu\nu}{}^{i} \neq 0$ then the A_μ cannot be removed everywhere by a choice of gauge, only at chosen points.

An important distinction between gravitation and standard Yang-Mills theories is that whilst the $A_\mu{}^{i}(x)$ are not directly observable, their counterparts, the $\{{}^{\mu}_{\nu\kappa}\}$ certainly are. They determine the inertial or gravitational forces acting on objects which aren't in free-fall and hence are responsible for objects acquiring a weight on planetary surfaces, cars skidding on corners, seasickness, etc. The 'true' unobservable gravitational potentials are the $g_{\mu\nu}(x)$ metric tensor components which certainly don't play an analogous role to the $A_\mu{}^{i}(x)$. This fact is at the heart of the problem of expressing metric theories of gravitation as gauge theories.

Another important distinction is the form of the actions for gravitation and for Yang-Mills theories.

The field equations for gravitation in matter-free space are,

$$\underline{R_{\mu\nu} \quad = \quad 0} \tag{8a}$$

where

$$R_{\mu\nu} = R_{\nu\mu} \equiv R^{\lambda}{}_{\mu\nu\lambda} = \partial_\nu \{{}^{\lambda}_{\mu\lambda}\} - \partial_\lambda \{{}^{\lambda}_{\mu\nu}\} + \{{}^{\kappa}_{\mu\lambda}\}\{{}^{\lambda}_{\kappa\nu}\} - \{{}^{\kappa}_{\mu\nu}\}\{{}^{\lambda}_{\kappa\lambda}\} \tag{8b}$$

is the symmetric Ricci tensor. These ten independent second order non-linear differential equations may be solved (in principle), up to constants of integration, for the ten potentials, $g_{\mu\nu}$. The action which yields these field equations is,

$$\underline{I_E \quad = \quad \int d^4x \sqrt{-g}\, R} \tag{9}$$

where, $R = g^{\mu\nu} R_{\mu\nu}$ and $g = \det g_{\mu\nu}$.

Varying I_E with respect to the independent fields, $g_{\mu\nu}$ gives (see Appendix C),

$$\frac{\delta I_E}{\delta g_{\mu\nu}} = \sqrt{-g}\,(\tfrac{1}{2}\,g^{\mu\nu}R - R^{\mu\nu}) .$$

In matter-free space the field equations imply that $R = 0$ and hence reduce to $R^{\mu\nu} = 0$, as required.

Notice that the Einstein action, I_E is linear in the curvature components (field strengths) $R^{\mu}{}_{\nu\kappa\lambda}$. The standard Yang-Mills action, on the other hand, is quadratic in the field strengths,

$$\underline{I_{Y\text{-}M} = \int d^4x \; F_{\mu\nu}{}^{i} \; F^{\mu\nu}{}_{i}} . \tag{10}$$

(The group indices, i are contracted using the Cartan-Killing metric[41] G_{ij} on the group manifold,

$$G_{ij} = \mathrm{Tr}(\mathrm{ad}(X_i)\mathrm{ad}(X_j)) = f_{ik}{}^{n}\, f_{jn}{}^{k}$$

where $\mathrm{ad}(X_i)$ is the adjoint matrix representation with elements,

$$(X_i)_j{}^k = f_{ij}{}^k).$$

In order to establish the gauge theoretical structure of gravitation, it is important to examine the coupling of gravity to matter fields. We henceforth abandon the picture of small test particles in otherwise matter-free space and with modern quantum field theory in mind, consider matter as Bose and Fermi fields in space-time.

2) Coupling of Matter to Gravity

Bose particles are associated with scalar, vector and in principle, tensor fields, although there are good theoretical and experimental reasons for believing that the only tensor particle in nature is the spin 2 graviton[73]. It is not difficult to see how these fields may couple to gravity. Consider an action Im for Bose fields, ϕ, written as,

$$Im = \int d^4x \, \mathcal{L}_m(\phi, \partial_\mu \phi) \; .$$

This is the standard form of field theory action on flat space-time expressed in terms of the convenient Lorentz coordinates. The same action may be written in curvilinear coordinates simply by making the replacements,

$$\eta_{\mu\nu} \longrightarrow g_{\mu\nu}(x)$$

$$\partial_\mu \longrightarrow \nabla_\mu \; , \quad \text{the covariant (physical) derivative,}$$

$$\text{for example,} \quad \nabla_\mu A^\lambda = \partial_\mu A^\lambda + \{{}^{\lambda}_{\mu\nu}\} A^\nu$$

and

$$d^4x \longrightarrow \sqrt{-g}\, d^4x, \quad \text{where } \sqrt{-g} \text{ is a Jacobian factor}$$

(- see Appendix C).

Hence our flat space-time action Im may be written in general coordinates as

$$Im = \int d^4x \, \sqrt{-g} \, \mathcal{L}_m(g_{\mu\nu}, \phi, \nabla_\mu \phi) \; .$$

A very simple example is provided by the flat space-time Lagrangian for the free massless scalar field written in Lorentz coordinates as

$$\mathcal{L}_m = \partial^\mu \phi \, \partial_\mu \phi \equiv \eta^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi$$

and in curvilinear coordinates this becomes,

$$\mathcal{L}_m = g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi \, .$$

Written in general curvilinear coordinates, an action I_m is also a suitable invariant action on a curved space-time. Hence the prescription for writing a flat space action in terms of curvilinear coordinates also couples in gravitation (since the $g_{\mu\nu}$ and $\{^{\mu}_{\nu\kappa}\}$ don't distinguish gravitational from inertial effects). Of course, once we have a gravitational field then the Einstein action must be added to I_m so that $g_{\mu\nu}(x)$ become dynamical fields. The complete action for Bose fields coupled to gravity is therefore,

$$I = \frac{1}{\kappa^2} \int d^4x \, \sqrt{-g} \, R + \int d^4x \, \sqrt{-g} \, \mathcal{L}_m(g_{\mu\nu}, \phi, \nabla_\mu \phi) .$$

Here we have introduced the gravitational coupling constant κ. We shall work in units where $c = \hbar = 1$, so that length = $(\text{mass})^{-1}$ and the requirement that I is dimensionless, $[I] = 0$, leads to the requirement that $[\mathcal{L}_m] = (\text{mass})^4$ and that $\left[\frac{R}{\kappa^2}\right] = (\text{mass})^4$. However $[R] = (\text{mass})^2$, since it is second order in derivatives, hence, $[\kappa^2] = (\text{mass})^{-2}$, i.e. $\underline{[\kappa] = (\text{mass})^{-1}}$. (The fact that the gravitational coupling constant is not dimensionless is significant in the quantum problem, see section II.1).

The field equations for gravity in the presence of Bosonic matter are therefore

$$\frac{\delta I}{\delta g_{\mu\nu}} = \frac{1}{\kappa^2} \sqrt{-g} \, (\tfrac{1}{2} g^{\mu\nu} R - R^{\mu\nu}) + \frac{\delta I_m}{\delta g_{\mu\nu}} = 0$$

or,

$$\underline{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad = \quad \kappa^2 \, T_{\mu\nu}} \tag{11}$$

where $\underline{T_{\mu\nu} \equiv \frac{1}{\sqrt{g}} \frac{\delta I_m}{\delta g^{\mu\nu}}}$ is the symmetric stress-energy tensor[74] of the matter fields, (see also section 4).

Fermionic matter is described by spinors or spinor valued fields which, by definition, form representations of the group SL(2,c). It is through the group homomorphism, SL(2,c) $\cong$ SO(1,3) that spinors also provide representations of the Lorentz group, SO(1,3). (See Appendix A for details of the spinor representations of the Lorentz group). The fundamental two component Weyl spinor representation and its conjugate describe spin $\frac{1}{2}$ particles. Dirac spinors are four component spinors which are the direct sum of the two fundamental representations and physically describe massive, charged, spin $\frac{1}{2}$ particles. Massless spin $\frac{1}{2}$ particles are described by either of the two fundamental representations which then correspond to the two helicities. Finally, neutral spin $\frac{1}{2}$ particles are described by self conjugate Dirac spinors, known as Majorana spinors. From the fundamental representations, all the finite dimensional representations of the Lorentz group may be constructed as the tensor (direct) product of n copies of the fundamental ones. The spin 0, 1, 2, ... scalar, vector and tensor representations are constructed from an even number direct product and the spin $\frac{1}{2}, \frac{3}{2}$, ... spinor representations from an odd number direct product.

There is an important difference, from our point of view, between the even (Bose) and odd (Fermi) representations of SO(1,3):

As we generalize the Lorentz group, SO(1,3) to the general linear group, GL(4, $\mathbb{R}$) then the vector and tensor representations become representations of $GL(4, \mathbb{R})$ but the spin $\frac{1}{2}$. $\frac{3}{2}$, ... representations cannot be generalized because they relate to the group $SO(1,3)$ only through the homomorphism $SL(2,c) \cong SO(1,3)$.

The prescription which we outlined for coupling Bose matter fields to gravity was based on the fact that vectors and tensors are not restricted to Lorentz coordinate systems but may be written in any curvilinear coordinates. In the presence of a gravitational field we cannot set up Lorentz coordinate frames on space-time and so it appears, cannot define the spinor fields describing Fermions. We must remember, however, that spinors and spinor fields are local objects in space-time and recall the equivalence principle which requires the existence of local Lorentz (inertial) frames at each point in space-time. The mathematical notion behind the physics of the equivalence principle was established by Cartan[6] in a series of papers in the 1920's. In this work Cartan developed what is now termed 'modern differential geometry' which has several features of importance to our discussion:

(a) Vectors on a manifold, M, act at a point as directional derivative operators thus giving the rate of change of functions on M in a specified direction. The aggregate of all such operators at any point forms a vector space, tangent to M at the point where these vectors operate. (The point itself is identified as the zero vector). In the case of the space-time manifold these tangent spaces have a Minkowski metric and are thus isomorphic to flat Minkowski space-time. It therefore follows that spinors may be defined at each point in space-time as representations of the Lorentz group acting in the tangent space. Hence they have the same relationship to the tangent space vectors as flat space-time spinors have to Lorentz vectors.

(b) It is important to recognize that the component formulation of general coordinate covariant A_μ and contravariant A^μ vectors is contained within the operator definition. The advantage of the notion of a tangent space is that it allows for more general, non-coordinate based vectors and in particular, spinors which cannot be coordinate based except on flat space-time.

(c) Of great importance to the identification of the gauge theoretical structure of gravity is Cartan's definition of a connection on a manifold, based on the action at each point, of a Lie symmetry group G. This Cartan connection, Γ is defined in a certain class of manifolds, now known as 'fibre bundles with a Cartan connection'. More general fibre bundles over space-time are known to be the manifolds upon which Yang-Mills gauge theories have a geometrical interpretation[(19),(56)]. Fibre bundles with a Cartan connection are the type which define a connection (a parallel transport process) on the base (space-time) manifold. Hence we anticipate that a gauge theory of gravity will be incomplete unless it is one with a Cartan connection.

An appendix dealing with fibre bundles and modern differential geometry has been added to the end of this chapter and will henceforth be referred to simply as 'the appendix'. We now go on to examine the impact that Cartan's 'new geometry' has on Einstein's physical theory of gravitation.

3) Einstein-Cartan Gravity - The Inclusion of Spin

The new feature in the Cartan geometry is a more general connection, with coordinate based components $\Gamma_{\nu\kappa}{}^{\mu}$ which replaces the Christoffel connection $\{{}^{\mu}_{\nu\kappa}\}$. Now, $\Gamma_{\nu\kappa}{}^{\mu}$ is not a tensor but transforms just like $\{{}^{\mu}_{\nu\kappa}\}$, according to (4). It therefore follows that its antisymmetric part,

$$\underline{S_{\mu\nu}{}^{\kappa} = \Gamma_{\mu\nu}{}^{\kappa} - \Gamma_{\nu\mu}{}^{\kappa} \equiv \Gamma_{[\mu\nu]}{}^{\kappa}} \tag{12}$$

is a tensor and is known as the torsion tensor field.

An important physical constraint on the differential geometry of space-time is the metric condition[3] that the metric tensor is covariantly constant,

$$\underline{\nabla_{\kappa} g_{\mu\nu} = \partial_{\kappa} g_{\mu\nu} - \Gamma_{\kappa\mu}{}^{\lambda} g_{\lambda\nu} - \Gamma_{\kappa\nu}{}^{\lambda} g_{\mu\lambda} = 0}\,, \tag{13}$$

(so that parallel displacement commutes with the raising and lowering of indices). This equation may be solved for the $\Gamma_{\mu\nu}{}^{\kappa}$ to obtain,

$$\underline{\Gamma_{\mu\nu}{}^{\kappa} = \{{}^{\kappa}_{\mu\nu}\} + \tfrac{1}{2} K_{\mu\nu}{}^{\kappa}} \tag{14a}$$

where,

$$\underline{K_{\mu\nu}{}^{\kappa} = S_{\mu\nu}{}^{\kappa} - S_{\nu}{}^{\kappa}{}_{\mu} + S^{\kappa}{}_{\mu\nu} = - K_{\mu}{}^{\kappa}{}_{\nu}}. \tag{14b}$$

The Riemann curvature tensor is defined by the same integrability condition on $\Gamma_{\mu\nu}{}^{\kappa}$ as that previously on $\{{}^{\kappa}_{\mu\nu}\}$ and therefore takes the form,

$$\underline{R^{\mu}{}_{\nu\kappa\lambda} = \partial_{\kappa}\Gamma_{\lambda\nu}{}^{\mu} - \partial_{\lambda}\Gamma_{\kappa\nu}{}^{\mu} + \Gamma_{\lambda\nu}{}^{\sigma}\Gamma_{\kappa\sigma}{}^{\mu} - \Gamma_{\kappa\nu}{}^{\sigma}\Gamma_{\lambda\sigma}{}^{\mu}} \tag{15}$$

The Ricci tensor, $R_{\mu\nu} = R^{\lambda}{}_{\mu\nu\lambda}$ is no longer symmetric, $R_{\mu\nu} \neq R_{\nu\mu}$, since $\Gamma_{\mu\nu}{}^{\kappa}$ is not symmetric. However, the curvature scalar, $R = g^{\mu\nu}R_{\mu\nu}$ only depends on the symmetric part of $R_{\mu\nu}$ and the gravitational field equations, in the absence of matter

$$\frac{\delta I_E}{\delta g_{\mu\nu}} = 0, \quad \text{now yield,} \quad R_{\mu\nu} + R_{\nu\mu} = 0.$$

There are still only ten field equations which may be solved for the $g_{\mu\nu}$ and hence the $\{{\kappa \atop \mu\nu}\}$ but Einstein gravity has no information about torsion in matter-free space. In a space-time with torsion the $g_{\mu\nu}$ are no longer the only independent fields, to these must be added the components of the torsion tensor field, $S_{\mu\nu}{}^{\kappa} = \Gamma_{[\mu\nu]}{}^{\kappa}$. Approaching gravity theory simply from the point of view of the Einstein action, we can treat all the $\Gamma_{\mu\nu}{}^{\kappa}$ as well as the $g_{\mu\nu}$ as independent fields. The field equations for the $\Gamma_{\mu\nu}{}^{\kappa}$ however simply yield[52] $\Gamma_{\mu\nu}{}^{\kappa} = \{{\kappa \atop \mu\nu}\}$, in the absence of matter (see also Appendix C) so that there is no torsion and the $\Gamma_{\mu\nu}{}^{\kappa}$ are determined from the $g_{\mu\nu}$. It was Cartan himself who first suggested that torsion might be generated by the intrinsic angular momentum of matter and only exist inside matter distributions. Later on, Weyl[78] showed that when gravity was coupled to a Dirac electron field then the affine connection components were no longer symmetric. Finally Kibble[47] and (independently) Sciama[67] showed that the non-symmetric part of the connection (i.e. the torsion) was generated by the local intrinsic spin angular momentum tensor of the matter fields. The gravitational field equations now become,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu} \tag{16a}$$

$$S_{\mu\nu}{}^{\kappa} = \frac{\kappa^2}{2\sqrt{g}} \left(\mathfrak{m}_{\mu\nu}{}^{\kappa} + \frac{1}{2} g_{\mu}{}^{\kappa} \mathfrak{m}_{\nu}{}^{\lambda}{}_{\lambda} - \frac{1}{2} g_{\nu}{}^{\kappa} \mathfrak{m}_{\mu}{}^{\lambda}{}_{\lambda} \right) \tag{16b}$$

where $\mathfrak{m}_{\mu\nu}{}^{\kappa}$ is the spin angular momentum density (see next section) and $R_{\mu\nu}$ and $T_{\mu\nu}$ are non-symmetric in general.

In establishing and understanding these field equations we must be able to write down spinor Lagrangians in curved space-time. The employment of the equivalence principle to do this leads automatically to the consideration of local Lorentz frames and hence, local Lorentz invariance[(5),(40),(43),(47),(67),(71)].

4) A Lorentz Gauge Theory

First we consider an action, I_m for spinorial matter fields, ψ on flat Minkowski space-time,

$$I_m = \int d^4x \, \mathcal{L}_m(\psi, \partial_a \psi).$$

Here $\partial_a \equiv \dfrac{\partial}{\partial x^a}$ and x^a are Lorentz coordinates on space time, (we shall henceforth use the Latin indices $a, b, c, d \ldots$ for Lorentz coordinates and the Greek indices $\mu, \nu, \kappa, \lambda \ldots$ for general curvilinear coordinates). The requirement that I_m is a Lorentz invariant reads

$$\delta I_m \equiv \int d^4x' \mathcal{L}_m' - \int d^4x \mathcal{L}_m = 0,$$

under the infinitesimal transformations of the inhomogeneous Lorentz group (Poincaré group),

$$\delta x^a = \omega^a{}_b x^b + \xi^a$$

and

$$\delta\psi \equiv \psi'(x') - \psi(x) = \frac{i}{2} \omega^{ab} M_{ab} \psi(x),$$

where $M_{ab} = -M_{ba}$ are the generators of SO(1,3).

We see that, quite generally,

$$\delta I_m = \int d^4x \left[\frac{\partial\mathcal{L}_m}{\partial\psi} - \partial_a\left(\frac{\partial\mathcal{L}_m}{\partial\partial_a\psi}\right)\right] \delta_o\psi + \oint d\mathcal{S}_a \left[\delta x^a \mathcal{L}_m + \frac{\partial\mathcal{L}_m}{\partial\partial_a\psi}\,\delta_o\psi\right]$$

where $\delta_o\psi \equiv \psi'(x) - \psi(x) = \delta\psi - \delta x^a\partial_a\psi$.

The volume term in δI_m is the variation of the action which vanishes identically on-shell. The surface term may be rewritten,

$$\oint ds_a \left[(\delta^a{}_b\mathcal{L}_m - (\partial_b\psi)\,\frac{\partial\mathcal{L}_m}{\partial\partial_a\psi})(\omega^b{}_c x^c + \xi^b) + \frac{\partial\mathcal{L}_m}{\partial\partial_a\psi}\,\frac{i}{2}\,\omega^{ab}\,M_{ab}\psi\right] = 0$$

and we separate the coefficients of the parameters ω^{ab} and ξ^b to obtain the conservation laws:

(a) For ξ^b,
$$\int_{t_1} ds_a\, T^{ab} = \int_{t_2} ds_a\, T^{ab}$$

where

$$\underline{T^{ab} = \eta^{ab}\mathcal{L}_m - \frac{\partial\mathcal{L}_m}{\partial\partial_a\psi}\,\partial^b\psi} \qquad (17)$$

and t_1, t_2 refer to different spacelike hypersurfaces, extending to spatial infinity.

(b) For ω^{ab},
$$\int_{t_1} ds_a\, J^{abc} = \int_{t_2} ds_a\, J^{abc}$$

where

$$J^{abc} = (T^{ab}x^c - T^{ac}x^b) + \frac{\partial\mathcal{L}_m}{\partial\partial_a\psi}\, i\, M^{bc}\psi \quad . \qquad (18)$$

The tensor T^{ab} given by (17) is the canonical energy-momentum-stress tensor, satisfying the local conservation law,

$$\partial_a T^{ab} = 0 .$$

The tensor J^{abc} is the spin-angular momentum density, satisfying the local conservation law,

$$\partial_a J^{abc} = T^{cb} - T^{bc} + \partial_a \mathcal{M}^{abc} = 0$$

where $$\underline{\mathcal{M}^{abc} \equiv \frac{\partial \mathcal{L}_m}{\partial \partial_a \psi}(i\, M^{bc} \psi)} \qquad (19)$$

is the intrinsic spin density. Clearly the conservation law for angular momentum may be written,

$$\partial_a \mathcal{M}^{abc} = T^{[bc]} .$$

So far we have considered I_m, expressed in terms of Lorentz coordinates x^a. On a flat Minkowski space-time these coordinates x^a label each point and specify a global inertial reference frame. More general curvilinear coordinates x^μ only serve to label points (events) in space-time and contain no information about the physical inertial frames. Hence if we are to consider a Lorentz invariant theory in general covariant form we must describe the physics in terms of:

(a) General curvilinear coordinates x^μ to label each point, P on space-time.

(b) Local Lorentz frames defined at each point P, labelled by $x^\mu(P)$, and required to define spinor fields at P. These Lorentz frames may be parametrized by coordinates x^a but it must be understood that these coordinates label points in a tangent space which coincides with the space-time manifold only if it is flat. Generally the only point of coincidence is P where $x^a = 0$ and $x^\mu = x^\mu(P)$.

The problem of understanding what is meant by the covariant form of $\partial_a\psi$ where ψ is a spinor field is best tackled by first establishing the relationship between ∂_a and ∂_μ at any given point and then clarifying the definition of the physical, or covariant derivative $\nabla_\mu\psi$ of a spinor field.

In fact the relationship between ∂_a and ∂_μ is easily understood using the modern differential geometric definition of vectors. As we have already stated, a vector V at a point P on a manifold may be viewed as a directional derivative operator acting at P and is an element of the vector space tangent to the manifold at P. A natural basis in which to express any vector in this space is the coordinate basis, $e_\mu = \dfrac{\partial}{\partial x^\mu}$ and clearly we may write $V = V^\mu e_\mu = V^\mu \partial_\mu$.

This coordinate basis transforms to any other, more general basis, e_i (i,j,k, ... will label any general basis henceforth) under the action of the general linear group, $GL(4, \mathbb{R})$. We write this transformation as,

$$e_i = (G)_i{}^\mu \, e_\mu$$

where G is a 4×4 matrix in the fundamental representation of $GL(4, \mathbb{R})$.

Now consider in particular the transformation from a coordinate basis e_μ to a local Lorentz basis e_a in which, by definition each of the e_o, e_1, e_2, e_3 acts along the relevant axis of a local Lorentz frame at P. This transformation is

$$e_a = (G)_a{}^\mu \, e_\mu$$

and has the inverse

$$e_\mu = (G^{-1})_\mu{}^a \, e_a.$$

The Riemann metric tensor $\underline{g}$ is a bilinear form (see the appendix) with components $g_{\mu\nu}$ or $g_{ab} = \eta_{ab}$ defined by

$$\underline{g}(e_\mu, e_\nu) = g_{\mu\nu}$$

or

$$\underline{g}(e_a, e_b) = g_{ab} = \eta_{ab} = \mathrm{diag}(+1, -1, -1, -1).$$

Hence,

$$\eta_{ab} = \underline{g}(e_a, e_b) = \underline{g}(G_a{}^\mu e_\mu, G_b{}^\nu e_\nu)$$

$$= G_a{}^\mu G_b{}^\nu \underline{g}(e_\mu, e_\nu)$$

i.e.

$$\underline{G_a{}^\mu G_b{}^\nu g_{\mu\nu} = \eta_{ab}} . \tag{20a}$$

Similarly,

$$\underline{(G^{-1})_\mu{}^a (G^{-1})_\nu{}^b \eta_{ab} = g_{\mu\nu}} . \tag{20b}$$

Either of these equations impose 10 constraints on the 16 $(G)_a^\mu$, leaving only 6 degrees of freedom, representing the freedom to pick any local Lorentz frame at the point P. In other words, for any given coordinate system, x^μ, the $G_a{}^\mu$ are determined up to the 6 parameter Lorentz group, $SO(1,3) \subset GL(4, R)$ under which, η_{ab} is by definition invariant.

Our discussion so far refers to the basis of a vector space tangent to a given point in space-time. A vector field $V(x)$ is a choice of vector, V at each point $x^\mu(P)$ in space-time. In terms of a coordinate basis, $V(x) = V^\mu(x)\partial_\mu$ where $V^\mu(x)$ are a set of four functions and the four ∂_μ form a basis at each point in space-time. From the coordinate basis we may transform to a local Lorentz basis $e_a(x)$ using local valued elements $(G(x))_a^\mu$ of $GL(4, \mathbb{R})$,

$$e_a(x) = (G(x))_a^\mu \partial_\mu .$$

The sixteen functions $G(x)_a{}^\mu$ are called the vierbein fields or tetrads. They establish an inertial frame at each point in space-time. We shall use the notation,

$$h^\mu{}_a(x) \equiv (G(x))_a^\mu$$

and that

$$h_\mu{}^a(x) \equiv (G^{-1}(x))_\mu^a \quad .$$

Equations (20) may now be written,

$$\underline{h^\mu{}_a(x)\ h^\nu{}_b(x)\ g_{\mu\nu}(x) \quad = \quad \eta_{ab}} \tag{21a}$$

and

$$\underline{h_\mu{}^a(x)\ h_\nu{}^b(x)\ \eta_{ab} \quad = \quad g_{\mu\nu}(x)} \quad . \tag{21b}$$

The vierbein fields are, for a given coordinate system x^μ, determined by equations (21) up to elements $(G(x))_a{}^b \equiv \Lambda(x)_a{}^b$ of the Lorentz group ($\Lambda_a{}^b$ is the 4×4 fundamental matrix representation of SO(1,3)) under which they transform as

$$\underline{h_\mu{}'^a(x) \quad = \quad \Lambda(x)^a{}_b\ h_\mu{}^b(x)} \quad . \tag{22a}$$

Under general coordinate transformations, $x^\mu \rightarrow x'^\mu \quad = \quad x'^\mu(x^\nu)$, so that $\partial_\mu \rightarrow \partial_\mu{}' \ = \ G_\mu{}^\nu \partial_\nu$ and hence by the chain rule $G_\mu{}^\nu \ = \ \dfrac{\partial x^\nu}{\partial x'^\mu}$, we see that by the group multiplication of GL(4, $\mathbb{R}$), the $h_\mu{}^a$ transforms as

$$\underline{h_\mu{}'^a(x) \quad = \quad \frac{\partial x^\nu}{\partial x'^\mu}\ h_\nu{}^a(x)} \quad . \tag{22b}$$

We can now summarize by stating that the Lorentz derivatives, ∂_a form a basis at each point on the space-time manifold, curved or flat, and are related to the coordinate basis by,

$$\partial_a = h^\mu{}_a(x)\ \partial_\mu \quad .$$

The covariant derivative of vectors and spinors on space-time may be described in terms of local Lorentz rotations under parallel transport if we refer these objects to the local Lorentz bases. Since these local Lorentz transformations act only in the tangent space they are not space-time transformations and we may employ the standard Yang-Mills gauge theory of SO(1,3) in order to define our covariant derivatives. A Dirac spinor field $\psi(x^\mu)$ will transform under local Lorentz transformations according to,

$$\underline{\psi(x^\mu) \rightarrow \psi'(x^\mu) = \exp(-\tfrac{i}{4}\omega^{ab}\sigma_{ab})\psi(x^\mu)}$$
$$\equiv \underline{S(\omega)\psi(x^\mu)} \tag{23}$$

where $-\frac{1}{2}\sigma_{ab} = -\frac{i}{4}[\gamma_a, \gamma_b]$ form the 4×4 Dirac matrix representation of the generators, M_{ab}, of SO(1,3). The Lorentz covariant derivative operator D_μ is then defined by

$$\underline{D_\mu = \partial_\mu - B_\mu = \partial_\mu - \tfrac{i}{2} B_\mu{}^{ab} M_{ab}} \tag{24}$$

and the transformation of the gauge potentials $B_\mu{}^{ab}$ is determined by the requirement that $D_\mu\psi$ transforms according to (23). We therefore find that,

$$\underline{B_\mu \rightarrow B_\mu' = S(\omega) B_\mu S(\omega)^{-1} - S(\omega)\, \partial_\mu S(\omega)^{-1}} \tag{25}$$

which is the standard form of the transformation of Yang-Mills gauge potentials under local gauge transformations. The gauge potentials and their transformation laws (25) may be established, (without any reference to associated matter fields such as $\psi(x^\mu)$) as geometrical properties of a manifold known as the principal fibre bundle over space-time, with structure group SO(1,3). This development is left to the appendix but we should note here that the geometrical picture yields the notion of the parallel transport of local (tangent) Lorentz frames through space-time.

With this standard Lorentz gauge theory, we see that a globally Lorentz invariant Lagrangian, $\mathcal{L}(\psi, \partial_a\psi)$ is made locally invariant through the replacement,

$$\partial_a \longrightarrow D_a = h^\mu{}_a D_\mu = h^\mu{}_a(\partial_\mu - B_\mu).$$

It is interesting to note that the relation $D_a = h^\mu{}_a D_\mu$ was used by Kibble[47] to introduce the fields $h^\mu{}_a(x)$. Our point of view is that the $h^\mu{}_a$ have already been defined and that $D_a = h^\mu{}_a D_\mu$ follows automatically from $\partial_a = h^\mu{}_a \partial_\mu$ since D_μ is the 'horizontal lift' of ∂_μ in the geometrical picture and this horizontal lift operation is linear for any vector, tangent to the base manifold. (See the appendix, page 78 for more details). A locally Lorentz invariant action is therefore expressed in terms of general coordinates as

$$I_m = \int d^4x \sqrt{-g}\, \mathcal{L}_m(\psi, D_a \psi).$$

The metric tensor is determined from the vierbein fields according to equations (21) and we see in particular that,

$$\underline{\sqrt{-g} = \pm \det h_{\mu a} \equiv \pm h} \tag{26}$$

so that

$$I_m = \int d^4x\, h\, \mathcal{L}_m(\psi, D_a\psi) \equiv \int d^4x\, h\, \bar{\mathcal{L}}_m (\psi, \partial_\mu\psi, h_{\mu a}, B_\mu{}^{ab}). \tag{27}$$

This is the form we expect an action for spinorial matter coupled to gravity to take. In fact this Lorentz gauge theory approach should also couple Bosonic matter to gravity which as we have already seen may be described by actions of the form,

$$I_m = \int d^4x\, h\, \mathcal{L}_m(\phi, \nabla_\mu\phi, g_{\mu\nu}) \equiv \int d^4x\, h\, \bar{\mathcal{L}}_m(\phi, \partial_\mu\phi, g_{\mu\nu}, \Gamma_{\mu\nu}{}^{\kappa}).$$

To relate this to (27) we require the relation between the potentials $B_\mu{}^{ab}$ and the components, $\Gamma_{\mu\nu}{}^{\kappa}$ of the affine connection. Now in the appendix we show how the affine connection components are gauge potentials $\Gamma_{\mu i}{}^{j}$ transforming under a local gauge transformation $(G(x))_i{}^j$ of the gauge group $GL(4, \mathbb{R})$ as

$$\underline{\Gamma_i{}^j \longrightarrow \Gamma'_i{}^j = (G\,\Gamma\,G^{-1})_i{}^j - (G\,d\,G^{-1})_i{}^j}$$
$$\equiv \underline{G_i{}^m\,(G^{-1})_n{}^j\,\Gamma_m{}^n - G_i{}^m\,d(G^{-1})_m{}^j} \qquad (28)$$

where $\Gamma_i{}^j \equiv dx^\mu\,\Gamma_{\mu i}{}^j$,

dx^μ is a coordinate basis for the dual vector space of differential 1-forms tangent to each point on the space-time manifold (see the appendix),

$df = (\partial_\mu f)dx^\mu$ for any function f.

In particular, for transformations between coordinate bases where the $GL(4, \mathbb{R})$ elements are restricted to,

$$(G(x))_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu}, \qquad (G^{-1}(x))_\mu{}^\nu = \frac{\partial x'^\nu}{\partial x^\mu}$$

then the transformations for the components $\Gamma_{\mu\nu}{}^{\kappa}$ take the familiar form of equations (4). We are more interested however in the transformations from general coordinate to local Lorentz bases which are clearly written as,

$$\underline{\Gamma_b^a = h^\nu{}_b\,h_\kappa{}^a\,\Gamma_\nu{}^\kappa - h^\omega{}_b\,d\,h_\omega{}^a.} \qquad (29)$$

Restricted to Lorentz bases the Γ^a_b are 4×4 matrix valued Lorentz connection 1-forms, transforming under the Lorentz subgroup of $GL(4, \mathbb{R})$ as

$$\Gamma_a{}^b \longrightarrow \Gamma'_a{}^b = (\Lambda\,\Gamma\,\Lambda^{-1})_a{}^b - (\Lambda\, d\,\Lambda^{-1})_a{}^b$$

where $\Lambda_b{}^a \equiv G^a{}_b$ are the 4×4 Lorentz matrices (Appendix A). We therefore identify $\Gamma_a{}^b$ with the fundamental matrix representation of the $SO(1,3)$ connection 1-form,

$$B(x) \equiv dx^\mu\, B_\mu(x) \equiv dx^\mu\, \frac{i}{2}\, B_\mu{}^{ab}\, M_{ab} \equiv \frac{i}{2}\, B^{ab}\, M_{ab}\,.$$

The fundamental (vector) representation of the generators M_{ab} is

$$[M_{ab}]_d{}^c = i(\eta_{bd}\,\delta_a{}^c - \eta_{ad}\,\delta_b{}^c)$$

so we set

$$\Gamma_a{}^b = \frac{i}{2}\, B^{cd}\, [M_{cd}]_a{}^b = -B^b{}_a = -dx^\mu\, B_\mu{}^b{}_a\,.$$

Hence equation (29) now reads,

$$-B_\mu{}^a{}_b = h^\nu{}_b\, h_\kappa{}^a\, \Gamma_{\mu\nu}{}^\kappa - h^\omega{}_b\, \partial_\mu\, h_\omega{}^a$$

$$\therefore \quad \underline{B_\mu{}^{ab} = h^{\omega b}\, \partial_\mu\, h_\omega{}^a - h^{\nu b}\, h_\kappa{}^a\, \Gamma_{\mu\nu}{}^\kappa .} \tag{30}$$

This is the relation we have been seeking between the affine connection components and the $SO(1,3)$ gauge potentials (which are in fact the components of the affine connection in a local Lorentz basis). Equation (30) is easily inverted to give

$$\underline{\Gamma_{\mu\nu}{}^\kappa = h^\kappa{}_a\, \partial_\mu\, h^a_\nu - h^\kappa{}_a\, h_{\nu b}\, B_\mu{}^{ab} .} \tag{31}$$

Either of equations (30) or (31) yields,

$$\underline{\partial_\mu h_\nu{}^a - \Gamma_{\mu\nu}{}^\kappa h_\kappa{}^a - B_\mu{}^a{}_b h_\nu{}^b = 0} \tag{32}$$

and this relation states that the covariant derivative of the vierbein field is zero, $\nabla_\mu h_\nu{}^a = 0$. Covariant differentiation of general coordinate tensor quantities is carried out using $\Gamma_{\mu\nu}{}^\kappa$ and of Lorentz tensor and spinor quantities using $B_\mu{}^{ab}$. We have defined the Lorentz covariant derivative operator, $D_\mu = \partial_\mu - B_\mu$, which ignores general coordinate indices and may also define a general coordinate derivative operator Δ_μ which ignores Lorentz indices so that the full covariant derivative operator ∇_μ is written,

$$\underline{\nabla_\mu = D_\mu + (\Delta_\mu - \partial_\mu).} \tag{33}$$

An alternative and clearer approach to covariant differentiation is to recognise that all we require for this process is the Lorentz connection $B_\mu{}^{ab}$ and the vierbein fields $h_\mu{}^a$. The covariant derivative of world tensors is then approached as follows, for example for a world vector V^μ,

$$\begin{aligned}\nabla_\mu V^\nu &= \nabla_\mu h^\nu{}_a V^a = h^\nu{}_a \nabla_\mu V^a = h^\nu{}_a D_\mu V^a \\ &= h^\nu{}_a(\partial_\mu - B_\mu{}^a{}_b V^b) \ .\end{aligned}$$

This simple relationship between the covariant derivative of world and Lorentz quantities rests entirely on our result, (32), that the covariant derivative of the vierbein fields vanishes. It is interesting to note that in many previous approaches to this subject, the relation $\nabla_\mu h_\nu{}^a = 0$ has been treated as a postulate[(15), (47), (71)] whereas we have derived it from the required transformation properties of the affine connection.

We are now entirely satisfied with our action (27) for the minimal coupling of all matter fields to gravity and turn our attention to the Einstein action which we recall takes the form,

$$I_E = \frac{1}{\kappa^2}\int d^4x\,\sqrt{-g}\,g^{\mu\nu}\,R^{\lambda}{}_{\mu\nu\lambda} .$$

For consistency we wish to rewrite this action in terms of the vierbein and SO(1,3) connection fields. From equation (15) we identify

$$R^{\kappa}{}_{\lambda\mu\nu} = (\partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu + [\Gamma_\mu , \Gamma_\nu])_\lambda{}^{\kappa}$$

where $(\Gamma_\mu\Gamma_\nu)_\lambda{}^{\kappa} \equiv \Gamma_{\mu\omega}{}^{\kappa}\,\Gamma_{\nu\lambda}{}^{\omega}$.

But from (31),

$$\begin{aligned}(\Gamma_\mu)_\nu{}^{\kappa} &= h^{\kappa}{}_a\,\partial_\mu\,h_\nu{}^a - h^{\kappa}{}_a\,h_{\nu b}\,B_\mu{}^{ab}\\ &\equiv (h^{-1}\,\partial_\mu\,h)_\nu{}^{\kappa} - (h^{-1}\,B_\mu\,h)_\nu{}^{\kappa} .\end{aligned}$$

Substituting this into the above expression for $R^{\kappa}{}_{\lambda\mu\nu}$ we obtain

$$\begin{aligned}R^{\kappa}{}_{\lambda\mu\nu} &= -(h^{-1}(\partial_\mu\,B_\nu - \partial_\nu B_\mu - [B_\mu,B_\nu]\;h)_\lambda{}^{\kappa}\\ &= -\,h^{\kappa}{}_a\,h_{\lambda b}(\partial_\mu B_\nu{}^{ab} - \partial_\nu B_\mu{}^{ab} + B_\mu{}^{ac}B_\nu{}^{b}{}_c - B_\nu{}^{ac}B_\mu{}^{b}{}_c).\end{aligned}$$

The quantities in the brackets are the components of the SO(1,3) gauge field strengths, $R_{\mu\nu}{}^{ab}$, defined by

$$\underline{[D_\mu, D_\nu] = -\frac{i}{2}\,R_{\mu\nu}{}^{ab}\,M_{ab}} \tag{34}$$

so that we may write,

$$\underline{R^{\kappa}{}_{\lambda\mu\nu} = -\,h^{\kappa}{}_a\,h_{\lambda b}\,R_{\mu\nu}{}^{ab}} \tag{35}$$

where

$$R_{\mu\nu}{}^{ab} = \partial_\mu B_\nu{}^{ab} - \partial_\nu B_\mu{}^{ab} + B_\mu{}^{ac} B_\nu{}^{b}{}_{c} - B_\nu{}^{ac} B_\mu{}^{b}{}_{c} \quad . \tag{36}$$

From this we see that the curvature scalar may be written,

$$R = g^{\mu\nu} R^{\lambda}{}_{\mu\nu\lambda} = h^{\mu}{}_{a} h^{\nu}{}_{b} R_{\mu\nu}{}^{ab}$$

and the Einstein action takes the form,

$$I_E = \frac{1}{\kappa^2}\int d^4x \sqrt{-g} = \frac{1}{\kappa^2}\int d^4x \, h \, h^{\mu}{}_{a} h^{\nu}{}_{b} R_{\mu\nu}{}^{ab} \quad . \tag{37}$$

The full action for the interaction of gravity with spinning matter is therefore

$$I = I_E + I_m = \frac{1}{\kappa^2}\int d^4x \, h \, h^{\mu}{}_{a} h^{\nu}{}_{b} R_{\mu\nu}{}^{ab} + \int d^4x \, h \, \mathcal{L}_m(\psi, D_a\psi) \tag{38}$$

'I' is a functional of the fields ψ, $h_\mu{}^a$ and $B_\mu{}^{ab}$ with the classical field equations

$$\frac{\delta I_m}{\delta\psi} = \frac{\delta I}{\delta h_{\mu a}} = \frac{\delta I}{\delta B_\mu{}^{ab}} = 0 \ .$$

The matter field equations depend on the explicit form of $\mathcal{L}_m$ and do not concern us in this chapter. The equations for the vierbein fields are

$$\frac{\delta I_E}{\delta h_{\mu a}} + \frac{\delta I_m}{\delta h_{\mu a}} = 0 \quad .$$

In Appendix C we evaluate the variation of the Einstein action and find that

$$\frac{\delta I_E}{\delta h^{\mu}{}_{a}} = 2h(R_\mu{}^a - \frac{1}{2} h_\mu{}^a R)$$

where $R_\mu{}^a \equiv h^\nu{}_b R_{\mu\nu}{}^{ab}$.

From the form of I_m in (38) we see that

$$\frac{\delta I_m}{\delta h^\mu{}_a} = \frac{\partial h}{\partial h^\mu{}_a} \mathcal{L}_m(\psi, D_a\psi) + h \frac{\partial \mathcal{L}_m}{\partial h^\mu{}_a}(\psi, D_a\psi)$$

but (Appendix C) $\dfrac{\partial h}{\partial h^\mu{}_a} = - h_\mu{}^a h$

and

$$\frac{\partial \mathcal{L}_m}{\partial h^\mu{}_a}(\psi, D_a\psi) = \frac{\partial \mathcal{L}_m}{\partial D_b\psi} \frac{\partial D_b\psi}{\partial h^\mu{}_a} = \frac{\partial \mathcal{L}_m}{\partial D_b\psi} \frac{\partial (h^\nu{}_b D_\nu\psi)}{\partial h^\mu{}_a}$$

$$= \frac{\partial \mathcal{L}_m}{\partial \partial_b\psi} \delta^a{}_b D_\mu \psi .$$

Therefore

$$\frac{\delta I_m}{\delta h^\mu{}_a} = - h\, h_\mu{}^a \mathcal{L}_m + h \frac{\partial \mathcal{L}_m}{\partial \partial_a\psi} h_\mu{}^b D_b\psi$$

and the field equations for the vierbeins are

$$\underline{R_\mu{}^a - \frac{1}{2} h_\mu{}^a R = \frac{\kappa^2}{2} (- \delta^a{}_b \mathcal{L}_m + \frac{\partial \mathcal{L}_m}{\partial \partial_a\psi} D_b\psi) h_\mu{}^b} . \qquad (39)$$

Multiplying through by $h_{\nu a}$ gives

$$\underline{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\kappa^2}{2}(-g_{\mu\nu}\mathcal{L}_m + \frac{\partial \mathcal{L}_m}{\partial \partial_\nu\psi} D_\mu\psi) \equiv \kappa^2 T_{\mu\nu}} . \qquad (40)$$

These are the sixteen non-symmetric Einstein field equations quoted in equation (16a). The source of the gravitational field is the non-symmetric stress-energy tensor $T_{\mu\nu}$ which we see is the covariant form of the canonical energy-momentum-stress tensor, T_{ab}, of equation (17).

The field equations for the $B_\mu{}^{ab}$ are

$$\frac{\delta I_E}{\delta B_\mu{}^{ab}} + \frac{\delta I_m}{\delta B_\mu{}^{ab}} = 0$$

and in Appendix C we find that

$$\frac{\delta I_E}{\delta B_\mu{}^{ab}} = \frac{h}{\kappa^2} (S_{ab}{}^\mu - h^\mu{}_a S_{\nu b}{}^\nu + h^\mu{}_b S_{\nu a}{}^\nu)$$

where $S_{\mu\nu}{}^\kappa$ are the components of the torsion tensor field,

$$\underline{S_{\mu\nu}{}^a \equiv h_\kappa{}^a S_{\mu\nu}{}^\kappa = D_\mu h_\nu{}^a - D_\nu h_\mu{}^a .} \tag{41}$$

This result follows directly from the definition (12) and the transformations (31) and is discussed further in Appendix C. The variation of the matter action gives

$$\frac{\delta I_m}{\delta B_\mu{}^{ab}} = h \frac{\partial \mathcal{L}_m}{\partial B_\mu{}^{ab}} = h \frac{\partial \mathcal{L}_m}{\partial D_c \psi} \frac{\partial D_c \psi}{\partial B_\mu{}^{ab}}$$

$$= h \frac{\partial \mathcal{L}_m}{\partial \partial_c \psi} (- \tfrac{i}{2} h^\mu{}_c M_{ab} \psi) .$$

Therefore

$$\frac{\delta I_m}{\delta B_\mu{}^{ab}} = - \tfrac{1}{2} h\, h^\mu{}_c \mathfrak{M}^c{}_{ab}$$

where $\mathfrak{M}^c{}_{ab} \equiv \dfrac{\partial \mathcal{L}_m}{\partial \partial_c \psi} (i M_{ab} \psi)$ is the intrinsic spin density (19).

Hence the field equations for the SO(1,3) gauge potentials are

$$\underline{S_{ab}{}^\mu - h^\mu{}_a S_{\nu b}{}^\nu + h^\mu{}_b S_{\nu a}{}^\nu = \frac{\kappa^2}{2} \mathfrak{M}^\mu{}_{ab}} . \tag{42}$$

Contracting the field equations (42) with $h_\mu{}^a$ we find,

$$S_{\nu b}{}^\nu = - \frac{\kappa^2}{4} \mathfrak{M}^\mu{}_{\mu b} = - \frac{\kappa^2}{4} \mathfrak{M}^a{}_{ab}$$

so that we may rewrite (42) as,

$$S_{ab}{}^{\mu} = \frac{\kappa^2}{2}\left(\mathcal{M}^{\mu}{}_{ab} - \frac{1}{2}h^{\mu}{}_{a}\mathcal{M}^{c}{}_{cb} + \frac{1}{2}h^{\mu}{}_{b}\mathcal{M}^{c}{}_{ca}\right)$$

or

$$\underline{S_{\mu\nu}{}^{a} = D_\mu h_\nu{}^a - D_\nu h_\mu{}^a = \frac{\kappa^2}{2}\left(\mathcal{M}^{a}{}_{\mu\nu} - \frac{1}{2}h_\mu{}^a\mathcal{M}^{c}{}_{c\nu} + \frac{1}{2}h_\nu{}^a\mathcal{M}^{c}{}_{c\mu}\right)} \quad (43)$$

These field equations for the $B_\mu{}^{ab}$ show that torsion is generated by the intrinsic spin angular momentum density of matter. The equations for the torsion or the $B_\mu{}^{ab}$ are algebraic showing that these fields do not propagate and will not therefore correspond to any physical particle states in nature. The role of the field equations (43) is therefore one of imposing non-dynamical constraints on the geometry of space-time. All the dynamical degrees of freedom of the gravitational field are contained in the vierbein $h_\mu{}^a$ or the metric $g_{\mu\nu}$ (see also section II.1).

In matter free space equations (43) become,

$$S_{\mu\nu}{}^{a} = D_\mu h_\nu{}^a - D_\nu h_\mu{}^a = 0$$

i.e.

$$\partial_\mu h_\nu{}^a - \partial_\nu h_\mu{}^a - B_\mu{}^a{}_b\, h_\nu{}^b + B_\nu{}^a{}_b\, h_\mu{}^b = 0 .$$

or

$$B_\mu{}^a{}_\nu - B_\nu{}^a{}_\mu = \partial_\mu h_\nu{}^a - \partial_\nu h_\mu{}^a \equiv L_{\mu\nu}{}^a .$$

Rewriting this equation twice with a cyclic permutation of indices and subtracting the third from the sum of the first two we obtain,

$$2\,B_\mu{}^a{}_\nu = L_{\mu\nu}{}^a + L_\nu{}^a{}_\mu - L^a{}_{\mu\nu}$$

so that

$$\underline{B_\mu{}^{ab} = \frac{1}{2}h^{\nu b}(\partial_\mu h_\nu{}^a - \partial_\nu h_\mu{}^a) + \frac{1}{2}h^{\nu b}h^{\kappa a}(\partial_\nu h_\kappa{}^c)h_{\mu c} - (a \leftrightarrow b)} \quad (44)$$

This expression for the torsion-free spin connection may also be obtained from equation (30) when the torsion-free affine connection, $\Gamma_{\mu\nu}{}^{\kappa} = \{{}^{\kappa}_{\mu\nu}\}$ is inserted.

When matter fields, ψ, are present then the spin connection $B_\mu{}^{ab}$ may be written as

$$B_\mu{}^{ab} = B_\mu{}^{ab}(h) + B_\mu{}^{ab}(\psi)$$

where $B_\mu{}^{ab}(h)$ is the torsion-free connection given by (44) and $B{}^{ab}(\psi)$ depends explicitly on the spin density $\mathcal{M}^{abc}$, (see Appendix C for the specific example of the supergravity Lagrangian). The fact that the torsion tensor field is determined by the algebraic field equations (43) implies that even in the presence of matter fields with intrinsic spin it may be removed in favour of the spin densities $\mathcal{M}^{abc}$. Hence we may view the torsion-free Christoffel symbols $\{{}^{\kappa}_{\mu\nu}\}$ or their vierbein equivalents $B_\mu{}^{ab}(h)$ of equation (44), as the only geometrical part of the connection and rewrite the Einstein equations (40) as

$$R_{\mu\nu}(g, \{{}^{\kappa}_{\mu\nu}\}) - \frac{1}{2} g_{\mu\nu} R(g, \{{}^{\kappa}_{\mu\nu}\}) = \kappa^2 T_{\mu\nu}{}'(\psi,\partial\psi,h_\mu{}^a,\partial h_\mu{}^a, \mathcal{M}^{abc})$$

where $T_{\mu\nu}{}'$ is the combined stress-energy tensor[(40)], modified from $T_{\mu\nu}$ by the $\mathcal{M}^{abc}$ terms brought across from the left hand side of (40).

We conclude that torsion as a new geometrical property of space-time may be introduced into gravitation theory but the only physical effects it should have may be absorbed into a modified stress-energy tensor.

In one respect however this removal of torsion from the theory is unsatisfactory. Because we are employing the field equation (or constraint) (43) to obtain

$$B_\mu{}^{ab} = B_\mu{}^{ab}(\psi, \partial\psi, h_\mu{}^a, \partial h_\mu{}^a)$$

our use of Lorentz covariant derivatives to couple gravity to matter is no longer a minimal coupling prescription (since the connection components now depend on the matter fields ψ). Hence if we are to identify gravity as a gauge theory it would appear to be important to retain the torsion as an independent field.

5) Gravitation as a Gauge Theory

We have finally arrived at the point where we may examine the central issue of this chapter, namely the status of gravity as a classical gauge theory.

One point which has emerged in our study so far is that in a theory which takes spin into account, a local Lorentz symmetry is a necessity. It is for this reason that we shall accept that a gauge theory of the Lorentz group is an essential part of any theory of gravitation which reduces to Einstein's in matter-free space. This local Lorentz structure is not simply a formal construction but something of great importance in making sense of the physics of curved space-time. Lorentz coordinates in flat space-time correspond directly to physical intervals (or distance measures) and local Lorentz frames at some event in curved space-time take on the operational role of measuring devices for the physical processes in the locality of the event. Observers may always establish local Lorentz frames with their associated local 'light cones' determining the space-time region in which their futures lie.

We have also discussed the local GL(4, $\mathbb{R}$) symmetry which is implicit in the four dimensional formulation of physical laws. This symmetry is simply the freedom to choose any local basis $\{e_i\}$ $(i = 0,1,2,3)$ and includes general coordinate invariance as a special class of bases. The components $\Gamma^k_{\cdot ij}$ of the affine connection emerge as the gauge potentials (connection components) in a $GL(4, \mathbb{R})$ gauge theory (see the appendix). Gravity is not simply a gauge theory of $GL(4, \mathbb{R})$ however because the affine connection is constrained by the physical requirements;

(a) The 4-dimensional world is a metric space with a local Minkowski structure, (spatial and temporal intervals).

(b) The affine connection and the metric are related by the condition (13) that the metric is covariantly constant, $\nabla_\kappa g_{\mu\nu} = 0$. This condition ensures that scalar products and in particular the length of 4-vectors are unchanged under parallel transport so that distance and time scales are unaffected by the parallel transport process. It is interesting to note that Weyl[23] attempted to relax this metric condition and obtain a more general geometry than that of Einstein with the hope of incorporating electromagnetism into the geometrical theory. Einstein pointed out however that amongst other things, one consequence of this generalized geometry would be that a collection of atoms in a given locality would have differing natural time scales due to their differing world histories and so would not give sharp spectral lines.

We may therefore summarize our findings so far with the statement that gravity is a constrained gauge theory of $GL(4, \mathbb{R})$ or equivalently, that gravity is a gauge theory of the Lorentz group with the vierbein fields judiciously incorporated to obtain the required geometrical picture. The sixteen vierbein fields $h_\mu{}^a$ were introduced as elements of the fundamental 4×4 matrix representation of $GL(4, \mathbb{R})$ but constrained to satisfy the ten conditions, $h_\mu{}^a h_{\nu a} = g_{\mu\nu}$. Clearly the weak point in the development so far is the fact that these vierbein fields fit so badly into the gauge theory picture. The question which arises is whether the $h_\mu{}^a$ themselves may be regarded as gauge potentials.

Certainly a symmetry which we have not exploited so far is the translational invariance of flat space-time theory. This invariance implies the conservation of field energy-momentum, $P_a = \int_t ds^b T_{ab}$, as we discussed in the beginning of section 4. An attractive idea therefore is to extend the gauge group SO(1,3) to the Poincaré group ISO(1,3) and to identify the $h_\mu{}^a$ as the gauge potentials associated with the translation generators, P_a.

Translational invariance was first considered as a gauge symmetry by Kibble[47] whose approach to the problem we shall outline. We consider infinitesimal Poincaré transformations in flat space-time,

$$\underline{\delta x^a = \omega^a{}_b x^b + \xi^a} \quad . \tag{45}$$

The effect on field variables, $\psi(x)$ is described either by the passive transformations,

$$\underline{\delta\psi \equiv \psi'(x') - \psi(x) = \frac{i}{2}\omega^{ab} M_{ab}\, \psi(x)} \tag{46a}$$

or by the active transformations,

$$\begin{aligned}
\underline{\delta_o\psi \equiv \psi'(x) - \psi(x)} & \underline{= \delta\psi - (\delta x^a)\partial_a\psi} \\
& \underline{= \frac{i}{2}\omega^{ab} M_{ab}\psi - (\omega^a{}_b x^b + \xi^a)\partial_a\psi} \\
& \underline{= \frac{i}{2}\omega^{ab} M_{ab}\psi + \frac{i}{2}\omega^{ab}\left[-i(x_a\partial_b - x_b\partial_a)\right]\psi - \xi^a\partial_a\psi} \quad .
\end{aligned} \tag{46b}$$

In (46b) the generators of the transformations on the R.H.S. correspond to the intrinsic spin angular momentum, the orbital angular momentum and the linear momentum (translations) respectively. When the Poincaré symmetry is gauged, $\omega^{ab} \to \omega^{ab}(x)$ and $\xi^a \to \xi^a(x)$, then the last two terms in (46b) lose their distinct character and become a generalized, or local, translation term,

$$\sim \zeta^a(x)\ \partial_a\ \psi$$

where,

$$\zeta^a(x) \quad = \quad \omega^{ab}(x)x_b + \xi^a(x) \ ..$$

Clearly this type of transformation, $\delta x^a = \zeta^a(x)$ is an infinitesimal transformation from Lorentz coordinates to some general curvilinear coordinates, $x^\mu = \delta^\mu{}_a(x^a + \delta x^a)$. A gauge theory of the Poincaré group therefore adds to the local Lorentz transformations, local translations which, considered as passive transformations, correspond to infinitesimal general coordinate transformations. Kibble constructed a theory invariant under the local transformations parametrized by $\omega^{ab}(x)$ and $\zeta^\mu(x)$ by obtaining a covariant derivative, $D_a\psi$ which transformed under the local group action in the same way as $\partial_a\psi$ does under the global group action, namely,

$$\underline{\delta(D_a\psi) \quad = \quad \frac{i}{2}\,\omega^{cd}\,M_{cd}\,D_a\psi + \omega_a{}^b\,D_b\,\psi} \quad . \tag{47}$$

This covariant derivative must be constructed from $\partial_\mu\psi$ which transforms as

$$\underline{\delta(\partial_\mu\psi) \quad = \quad \frac{i}{2}\,\omega^{ab}\,M_{ab}\,\partial_\mu\psi + \frac{i}{2}(\partial_\mu\omega^{ab})M_{ab}\psi - (\partial_\mu\zeta^\nu)\partial_\nu\psi} \ . \tag{48}$$

As a first step to obtaining (47) from (48) we recognise that the inhomogeneous second term in (48) may be dealt with by the introduction of standard Lorentz covariant derivatives (24),

$$D_\mu = \partial_\mu - B_\mu = \partial_\mu - \frac{i}{2} B_\mu{}^{ab} M_{ab} .$$

We then see that,

$$\underline{\delta(D_\mu \psi) = \frac{i}{2} \omega^{ab} M_{ab} D_\mu \psi - (\partial_\mu \zeta^\nu) D_\nu \psi} \tag{49}$$

provided,

$$\underline{\delta B_\mu{}^{ab} = \partial_\mu \omega^{ab} + \omega^a{}_c B_\mu{}^{cb} + \omega^b{}_c B_\mu{}^{ac} - (\partial_\mu \zeta^\nu) B_\nu{}^{ab}} \tag{50}$$

which is the infinitesimal form of (25) together with an infinitesimal general coordinate transformation term. Now since (49) is homogeneous in $D_\mu \psi$ no further progress will be made by introducing more gauge fields with inhomogeneous transformation laws. In fact we recognize that (47) may be obtained from (49) through the introduction of the fields $h^\mu{}_a$ such that,

$$\underline{D_a \psi = h^\mu{}_a D_\mu \psi} \tag{51}$$

and where the $h^\mu{}_a$ are required to transform as

$$\underline{\delta h^\mu{}_a = \omega_a{}^b h^\mu{}_b + (\partial_\nu \zeta^\mu) h^\nu{}_a .} \tag{52}$$

The identification of the $h^\mu{}_a$ as the vierbein fields now leads to all our previous results in section 4 and in particular we see that the transformations (52) are the infinitesimal form of our earlier transformations (22). In fact what Kibble succeeded in doing was to recognise some of the general properties the vierbein fields must have in order to obtain a local Lorentz invariant theory, simply from the required covariance of the theory. In our own approach we obtained all the properties of the vierbein from geometrical requirements. Hence we see that Kibble's approach does not introduce the $h_{\mu a}$ as gauge fields and sheds no new light on this particular problem.

A more direct approach is to write down the standard Yang-Mills gauge theory of ISO(1,3). This group is the Poincaré group with the ten generators, $X_i \equiv \{M_{ab}, P_a\}$, satisfying the algebra,

$$\underline{[M_{ab}, M_{cd}] = i(\eta_{ac} M_{bd} + \eta_{bd} M_{ac} - \eta_{ad} M_{bc} - \eta_{bc} M_{ad})} \tag{53a}$$

$$\underline{[M_{ab}, P_c] = i(\eta_{ac} P_b - \eta_{bc} P_a)} \tag{53b}$$

$$\underline{[P_a, P_b] = 0 .} \tag{53c}$$

This algebra may be written,

$$[X_i, X_j] = f_{ij}{}^k X_k \equiv \tfrac{1}{2} f_{ij}{}^{ab} M_{ab} + f_{ij}{}^a P_a$$

where

$$\underline{f_{ab}{}^{ef}{}_{cd} = i(\eta_{ac} \delta^{[ef]}_{bd} + \eta_{bd} \delta^{[ef]}_{ac} - \eta_{ad} \delta^{[ef]}_{bc} - \eta_{bc} \delta^{ef}_{[ad]})} \tag{54a}$$

$$\underline{f_{ab}{}_c{}^d = - f_c{}_{ab}{}^d = i(\eta_{ac} \delta_b{}^d - \eta_{bc} \delta_a{}^d)} \tag{54b}$$

$$\underline{\text{all other } f_{ij}{}^k = 0 ,} \tag{54c}$$

$$(\delta^{[ab]}_{cd} = \delta_c{}^a \delta_d^b - \delta_c^b \delta_d^a) .$$

We construct covariant derivatives,

$$\underline{\nabla_\mu \equiv \partial_\mu - A_\mu = \partial_\mu - iA_\mu{}^i X_i \equiv \partial_\mu - \frac{i}{2} B_\mu{}^{ab} M_{ab} - ih_\mu{}^a P_a} \tag{55}$$

where the gauge potentials transform under an element,

$\exp(i\,\varepsilon^j X_j)$ of ISO(1,3) according to

$$A_\mu \longrightarrow A_\mu' = \exp(i\varepsilon^j X_j) A_\mu \exp(-i\varepsilon^j X_j) - \exp(i\varepsilon^j X_j)\partial_\mu \exp(-i\varepsilon^j X_j).$$

For infinitesimal $\varepsilon^j = \{\omega^{ab}, \xi^a\}$ these transformations read

$$\underline{\delta A_\mu{}^k = i\,\varepsilon^i A_\mu{}^j f_{ij}{}^k + \partial_\mu \varepsilon^k} \tag{56}$$

Using the relations (54) we see that (56) yields,

$$\underline{\delta B_\mu{}^{ab} = \partial_\mu \omega^{ab} + \omega^a{}_c B_\mu{}^{cb} + \omega^b{}_c B_\mu{}^{ac}} \tag{57a}$$

and

$$\underline{\delta h_\mu{}^a = \partial_\mu \xi^a - B_\mu{}^a{}_c \xi^c + \omega^a{}_c h_\mu{}^c} \tag{57b}$$

The potentials $B_\mu{}^{ab}$ and $h_\mu{}^a$ together with their transformation laws (57) are the basis for a Yang-Mills gauge theory of ISO(1,3). The question is whether or not this gauge theory may be identified as a theory of gravitation. The Lorentz part of the theory, with $\xi^a = 0$, yields the transformation laws which we required in section 4 and presents no new problems here. It is the translational, P_a generators in ISO(1,3) which we cannot properly accommodate. The infinitesimal transformation of the matter fields ψ under the translational part of the group is given by

$$\delta\psi = i\,\xi^a P_a \psi .$$

In flat space-time we know that $P_a = -i\,\partial_a$, so that

$$\delta\psi = \xi^a \partial_a \psi .$$

In curved space-time however the ∂_a no longer form a coordinate basis in the tangent space so that their Lie derivatives no longer vanish,

$$[\partial_a, \partial_b] \neq 0 \qquad \text{(in curved space-time)}$$

and clearly the identification $P_a = -i\partial_a$ contradicts the chosen algebra (53c) for the translation generators. The identification of the $h_\mu{}^a$, translational gauge potentials as the vierbein fields causes further problems. First, notice that the covariant derivative ∇_μ given by (55) collapses to $\nabla_\mu = -\frac{i}{2} B_\mu{}^{ab} M_{ab}$, a pure Lorentz rotation, (since $-i\,h_\mu{}^a P_a = -h_\mu{}^a \partial_a = -\partial_\mu$). Secondly, the inhomogeneous transformation of the 'vierbeins' $h_\mu{}^a$ given by (57b) has no geometrical basis at all, it is only the last term in (57b) that we require.

Clearly the interpretation of P_a as the generators of space-time translations and the vierbeins as the corresponding gauge fields must be treated with greater care. In Yang-Mills gauge theories the gauge group only effects internal transformations at a point in space-time. In our Lorentz gauge theory of the previous section we considered our local Lorentz transformations as transformations in the tangent space to space-time with the intrinsic spin angular momentum as the generators. It doesn't seem possible to regard the P_a generators as acting only in the tangent space since their interpretation as translation generators necessitates their role as vector operators which effect transformations on the space-time manifold itself. The interpretation of the vierbeins as gauge fields appears therefore inevitably to involve active space-time transformation symmetries. A related problem must surely be that whilst Yang-Mills gauge theories only have a geometrical interpretation in some extended fibre bundle space(12), gravitation may be described purely in terms of the space-time geometry.

To understand how the vierbein fields fit into a gauge theory it is important that we should recognise that only a certain class of gauge theories define a geometry on space-time and that these theories in fact are distinctive because they contain objects with the properties of the vierbein fields in addition to the usual connections. We close this chapter by clarifying the above statement, drawing from modern differential geometry and recent work(69),(77) on gravitation as a spontaneously broken gauge theory to produce a concise account of the theory of gravity upon which our later chapters will be based.

An affine geometry on an n-dimensional differentiable manifold M_n defines the notion of the curvature and torsion of M_n. More explicitly, the curvature and torsion may be defined by the Cartan structure equations,

$$\underline{d\theta^i + \Gamma^i_j \wedge \theta^j = S^i} \qquad (58a)$$

$$\underline{d\Gamma^i_j + \Gamma^i_k \wedge \Gamma^k_j = R^i_j} \qquad (58b)$$

($i,j,k = 1, \ldots, n$: $\omega \wedge \sigma$ is the exterior product of the differential forms ω and σ)

where: S^i is the torsion 2-form with components,

$S^i_{jk} = S^i(e_j, e_k)$ and $\{e_i\}$ form a basis for vector fields on M_n (see section 4).

$R^i_{\cdot j}$ is the curvature 2-form with components,

$$R^i_{jk\ell} = R^i_j\,(e_k,\ e_\ell).$$

$\Gamma^i_{\cdot j}$ is the connection 1-form (see equations (28))

and $$\Gamma^i_{jk} = \Gamma^i_k\,(e_j).$$

θ^i is the canonical 1-form, $\theta^i(e_j) = \delta^i_j$.

In the appendix to this chapter we show how $\Gamma^i_{\cdot j}$, θ^i and the structure equations may be defined in terms of a principal fibre bundle, $P(M_n,\ A(n,\ \mathbb{R}))$, over M_n with a Cartan connection. Here the structure group $A(n,\ \mathbb{R})$ is the affine group, the group of transformations on $\mathbb{R}^n$ consisting of all general linear, $GL(n,\ \mathbb{R})$, transformations about the origin and all translations on $\mathbb{R}^n$. Clearly $GL(n,\ \mathbb{R})$ is a subgroup of $A(n,\ \mathbb{R})$ and $\mathbb{R}^n$ is the coset space, $\frac{A(n,\ \mathbb{R})}{GL(n,\ \mathbb{R})}$. The Cartan connection defines the soldering of the associated bundle $E\ (M_n,\ \mathbb{R}^n,\ A(n,\ \mathbb{R}),\ P)$ to M_n and hence yields the group theoretical notion of an (affine) geometry on M_n itself. The $\mathbb{R}^n$ valued solder form on $P(M_n,\ GL(n,\ \mathbb{R}))$ is θ whose components θ^i constitute the canonical 1-form and the $g\ell(n,\ \mathbb{R})$ valued connection form on $P(M_n,\ GL(n,\ \mathbb{R}))$ is Γ whose components $\Gamma^i_{\cdot j}$ constitute the connection 1-form.

Space-time is a 4-dimensional affine manifold with a (pseudo) Riemannian metric structure. As we have discussed at some length, this metric structure is required to be locally Minkowski so that we may always find bases $\{e_a\}$ for which,

$$g(e_a,\ e_b) = \eta_{ab} \equiv \mathrm{diag}(1,\ -1,\ -1,\ -1).$$

Moreover we also require that the affine connection is constrained to give a covariantly constant metric. This additional metric structure gives the SO(1,3) Lorentz subgroup of GL(4, $\mathbb{R}$) a special role in describing the geometry of space-time. Parallel transport may be described by length preserving Lorentz rotations, parametrized by the spin connection components $B_\mu{}^{ab}$ which are the gauge potentials in a standard Yang-Mills SO(1,3) gauge theory.

In section 4 (page 28) we identified,

$$\underline{\Gamma^a_b = - dx^\mu B_\mu{}^a{}_b .} \tag{59}$$

Furthermore, since we require $\theta^a(e_b) = \delta^a{}_b$ and since, by definition the vierbein fields must satisfy, $e_b = h^\mu{}_b \, \partial_\mu$, then it follows that,

$$\underline{\theta^a = dx^\mu h_\mu{}^a .} \tag{60}$$

The first Cartan structure equation (58a) now becomes, in a coordinate basis,

$$d(dx^\mu h_\mu{}^a) - dx^\mu B_\mu{}^a{}_b \wedge dx^\nu h_\nu{}^b = S_{\mu\nu}{}^a \, dx^\mu \wedge dx^\nu .$$

Using the rules for exterior differentiation we identify

$$S_{\mu\nu}{}^a = \tfrac{1}{2}(\partial_\mu h_\nu{}^a - \partial_\nu h_\mu{}^a - B_\mu{}^a{}_b h_\nu{}^b + B_\nu{}^a{}_b h_\mu{}^b)$$

which (up to a factor of $\frac{1}{2}$) is our earlier definition (41) for the torsion tensor.

Similarly, from the second structure equation (58b) we recover the SO(1,3) curvature tensor (36), (again, up to a factor of $\frac{1}{2}$),

$$R_{\mu\nu}{}^{ab} = \tfrac{1}{2}(\partial_\mu B_\nu{}^{ab} - \partial_\nu B_\mu{}^{ab} + B_\mu{}^{ac} B_\nu{}^b{}_c - B_\nu{}^{ac} B_\mu{}^b{}_c) .$$

The vierbein fields are the components of the canonical 1-form expressed in a coordinate basis and of course, only emerge in this context for an affine manifold with a local Minkowski structure. The geometry of space-time should have a description in terms of a Cartan connection on a suitably constrained affine bundle over space-time. We may be quite certain that the subgroup structure $GL(4, \mathbb{R})$ of the affine bundle is reducible to $SO(1,3)$ for a bundle over space-time. The remaining question concerns the coset structure of the standard fibre of the bundle soldered to space-time by the Cartan connection. This coset space must have the same dimension as space-time and hence be the coset space of some 10-dimensional group with respect to the 6-dimensional Lorentz subgroup. We have three candidates for this 10-dimensional group:

(i) The Poincaré group $ISO(1,3)$ where the coset space $\frac{ISO(1,3)}{(SO(1,3)}$ is isomorphic to Minkowski space-time.

(ii) The anti de Sitter group $SO(2,3)$, where $\frac{SO(2,3)}{SO(1,3)} \cong$ anti de Sitter space-time[39].

(iii) The de Sitter group $SO(1,4)$, where $\frac{SO(1,4)}{SO(1,3)} \cong$ de Sitter space-time[39].

All three possibilities may define an affine geometry on space-time with a local Lorentz structure. We have examined the Poincaré group as a gauge group for gravitation and found a transformation law (57b) for the vierbein fields which we were unable to interpret. In fact the Poincaré group is related to $SO(2,3)$ and $SO(1,4)$ by the process of group contraction[33] and is a special limiting case of either of these pseudo-orthogonal groups. We shall return to this point later (see also section III.2) but now concentrate on the anti de Sitter group $SO(2,3)$, (the difference between $SO(2,3)$ and $SO(1,4)$ is only important when we consider spinor representations in Chapter III where we settle on SO(2,3)) .

The Lie group SO(2,3) has ten generators, $M_{AB} = - M_{BA}$ (AB = 0,1,2,3,5), satisfying the Lie algebra,

$$\underline{[M_{AB}, M_{CD}] = i(\eta_{AC} M_{BD} + \eta_{BD} M_{AC} - \eta_{AD} M_{BC} - \eta_{BC} M_{AD})} \tag{61}$$

where $\eta_{AB} = \mathrm{diag}(+1, -1, -1, -1, +1)$ is the metric in the 5-dimensional vector space which provides the fundamental representation of SO(2,3).

The M_{ab} (a,b = 0, 1, 2, 3) are the six generators of the Lorentz, SO(1,3), subgroup.

The M_{5a} are the four coset generators effecting 'translations' in the coset space, $\frac{SO(2,3)}{SO(1,3)}$.

We may construct an SO(2,3) covariant derivative,

$$\underline{\nabla_\mu = \partial_\mu - \frac{i}{2} B_\mu{}^{AB} M_{AB} \equiv \partial_\mu - \frac{i}{2} B_\mu{}^{ab} M_{ab} - imh_\mu{}^a M_{5a}} \tag{62}$$

where $mh_\mu{}^a \equiv B_\mu{}^{5a}$ and 'm' is a parameter with the dimensions of mass, included so that $h_\mu{}^a$ may be dimensionless. Under gauge transformations the $B_\mu{}^{ab}$ and $h_\mu{}^a$ both transform inhomogeneously and mix with one another according to,

$$\underline{\frac{i}{2} B_\mu{}^{AB} M_{AB} \longrightarrow \frac{i}{2} B_\mu{}'^{AB} M_{AB} = \exp(\tfrac{i}{2}\omega^{AB} M_{AB}) \tfrac{i}{2} B_\mu{}^{AB} M_{AB} \exp(-\tfrac{i}{2}\omega^{AB} M_{AB})}$$

$$- \underline{\exp(\tfrac{i}{2}\omega^{AB} M_{AB}) \partial_\mu \exp(-\tfrac{i}{2}\omega^{AB} M_{AB})} \tag{63}$$

where $\omega^{AB}(x)$ are the parameters of the gauge transformation. To interpret the $B_\mu{}^{ab}$ as the spin connection components and the $h_\mu{}^a$ as the vierbein fields however, we require them to transform according to equations (22a) and (25), namely,

$$\frac{i}{2}B'_\mu{}^{ab}M_{ab} = \exp(\frac{i}{2}\omega^{ab}M_{ab})\frac{i}{2}B_\mu{}^{ab}M_{ab}\exp(-\frac{i}{2}\omega^{ab}M_{ab})$$

$$- \exp(\frac{i}{2}\omega^{ab}M_{ab})\partial_\mu \exp(-\frac{i}{2}\omega^{ab}M_{ab})$$

and

$$h'_\mu{}^a = \Lambda(x)^a{}_b\, h_\mu{}^b \quad .$$

These transformations clearly correspond only to the SO(1,3) ($\omega^{5a} = 0$) part of the SO(2,3) gauge transformations (63). The only way that the full SO(2,3) group may act and yet preserve the required subgroup structure to the $B_\mu{}^{ab}$ and $h_\mu{}^a$ transformations is if the coset generators act non-linearly and induce (non-linear) subgroup transformations. This particular non-linear group action is in fact contained within the standard framework for the non-linear realization of a Lie group on its coset space with respect to some Lie subgroup. It has been known for some time that this non-linear realization of a group on its coset space provides an elegant framework for the description of field theories with spontaneously broken symmetries[(13),(64)]. The action for such a field theory is invariant under the full group but the ground or vacuum state is only invariant under the stability subgroup.

A gauge theory of SO(2,3), spontaneously broken to SO(1,3) was recently constructed by West and Stelle[(77),(69)]. They proposed an invariant gauge action with auxiliarly Higgs fields forming a vector multiplet $y^A(x)$ of SO(2,3). The symmetry is spontaneously broken to SO(1,3) by imposing the constraint that $y^A y_A = R^2$. In Appendix D we discuss this action further and show that, in the unitary gauge where $y^A(x) = (0,0,0,0,R)$, the action takes the form of the Einstein action together with a cosmological term

$\sim \frac{1}{R^2} \int d^4x \sqrt{-g}$. This suggests therefore that we may identify the $B_\mu{}^{ab}$ and $h_\mu{}^a$ as the spin connection components and the vierbein fields, provided we remain in the unitary gauge. Now the unitary gauge choice reduces the symmetry to SO(1,3) and so we see that the model of Stelle and West reinforces our observation that the $B_\mu{}^{ab}$ and $h_\mu{}^a$ can be identified up to the SO(1,3) subgroup transformations only. To understand how the full symmetry group operates in the gauge theory of SO(2,3) spontaneously broken to SO(1,3) it is convenient to employ the standard formalism for the non-linear realization of groups on coset spaces[13],[80]. In Appendix D we review this subject with particular regard to the group SO(2,3). Here we shall summarize the results of interest.

With $g \in SO(2,3)$, $h \in SO(1,3)$ and

$$e^{iy^a P_a} \in \frac{SO(2,3)}{SO(1,3)} \qquad (P_a \equiv m\, M_{5a})$$

then the group multiplication induces a non-linear realization of SO(2,3) upon $\frac{SO(2,3)}{SO(1,3)}$, given by,

$$\underline{g\, e^{iy^a P_a} = e^{iy'^a P_a} h_1} \quad . \tag{64}$$

Using the algebra (61) it is possible[36] to separate this equation and obtain explicitly the relations

$$\underline{y'^a = y'^a(g, y^a)} \tag{65a}$$

and

$$\underline{h_1 = h_1(g, y^a)} \quad . \tag{65b}$$

The y^a may be removed by a suitable gauge transformation ($g = e^{-iy^a P_a}$), they are coordinates on $\frac{SO(2,3)}{SO(1,3)}$ and play the role of the Goldstone modes in a field theory with a spontaneously broken symmetry, (see Appendix D and ref. (64)).

Here $h_1 \in SO(1,3)$, induced by the full $SO(2,3)$ action, (notice that for $g = h \in SO(1,3)$ then $h_1 = h$). In Appendix D we define non-linear fields $\bar{\phi}$ which transform under $g \in SO(2,3)$ only through $h_1(g,y^a) \in SO(1,3)$. In particular we define the non-linear $\bar{B}_\mu{}^{ab}(B_\mu{}^{ab}, h_\mu{}^a, y^a)$ and $\bar{h}_\mu{}^a(h_\mu{}^a, B_\mu{}^{ab}, y^a)$ which transform under $g \in SO(2,3)$ as

$$\underline{\frac{i}{2}\bar{B}_\mu{}^{ab}M_{ab} \longrightarrow \frac{i}{2}\bar{B}'_\mu{}^{ab}M_{ab} = h_1(g, y^a)\frac{i}{2}\bar{B}_\mu{}^{ab}\, h_1(g,y^a)^{-1} - h_1(g,y^a)\partial_\mu\, h_1(g,y^a)^{-1}} \tag{66a}$$

and

$$\underline{\bar{h}_\mu{}^a \longrightarrow \bar{h}'_\mu{}^a = \Lambda(g, y^a)^a{}_b\, \bar{h}^b} \tag{66b}$$

These relations show that $\bar{B}_\mu{}^{ab}$ and $\bar{h}_\mu{}^a$ transform in the required manner for the spin connection components and vierbein fields, not only under $SO(1,3)$ but the full (non-linear) action of $SO(2,3)$. The fields $\bar{B}_\mu{}^{ab}$ and $\bar{h}_\mu{}^a$ depend on the original $B_\mu{}^{ab}$ and $h_\mu{}^a$ and upon the 'Goldstone fields' $y^a(x)$ in a rather complicated manner (Appendix D). In the unitary gauge however where $y^a(x) = 0$, then

$$\bar{B}_\mu{}^{ab}(B_\mu{}^{ab}, h_\mu{}^a, y^a = 0) = B_\mu{}^{ab}$$

and

$$\bar{h}_\mu{}^a\,(h_\mu{}^a, B_\mu{}^{ab}, y^a = 0) = h_\mu{}^a\,.$$

Gauge transformations within the unitary gauge are restricted to $SO(1,3)$, so that $g = h = h_1$ and the realization is linear.

Gravitation may therefore be viewed as a spontaneously broken gauge theory of $SO(2,3)$ where the vierbein fields are the gauge potentials corresponding to the broken generators. The full $SO(2,3)$ symmetry of gravity is only realized in a highly non-linear manner but which becomes linear for the $SO(1,3)$ subgroup. This gauge theory has the geometrical description in terms of a Cartan connection on the anti de Sitter fibre bundle $P(M_4, SO(2,3))$ over space-time, M_4. The Cartan connection defines the soldering of the associated bundle $E(M_4, \frac{SO(2,3)}{SO(1,3)}, SO(2,3), P)$ to the space-time manifold. This bundle E with the anti de Sitter space $\frac{SO(2,3)}{SO(1,3)}$ as standard fibre may be pictured as space-time with a tangent anti de Sitter space at each point.

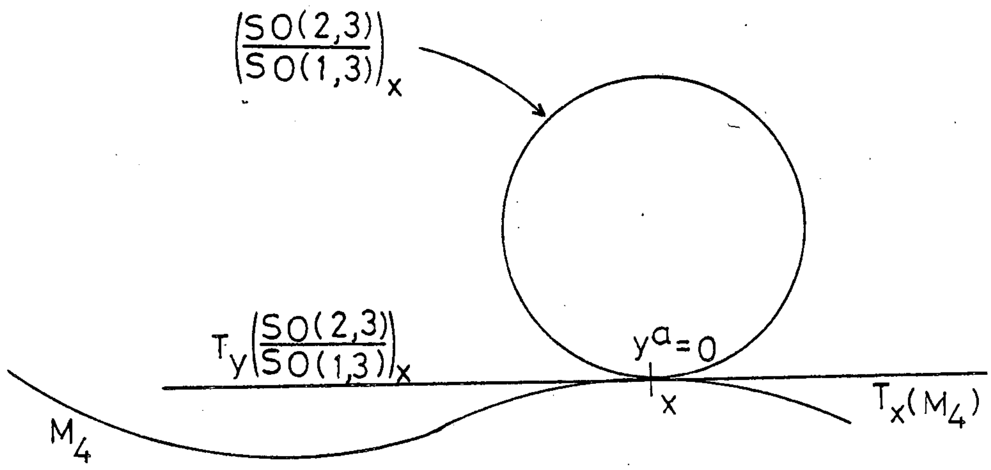


FIGURE 2

Figure 2 shows one fibre, $\left(\frac{SO(2,3)}{SO(1,3)}\right)_x$ of E, tangent to M_4 at the point x. The y^a are coordinates on the fibre and we see that a choice of Goldstone field $y^a(x)$ corresponds to a cross-section on E. The soldering of E to M_4 amounts to identifying the tangent space $T_x(M_4)$ to M_4 at x with the tangent space $T_y\left(\frac{SO(2,3)}{SO(1,3)}\right)_x$ to the fibre on the cross-section ($y^a = 0$ in Fig. 2), by an isomorphism. In the appendix we show how this isomorphism is defined by the form of soldering θ which from our preceding discussion we identify as,

$$\theta = dx^\mu \bar{e}_\mu{}^a P_a \qquad (P_a \equiv m M_{5a}) .$$

The choice of a cross-section $y^a(x)$ on E reduces the gauge symmetry to the stability subgroup $SO(1,3)$ which effects rigid rotations (Lorentz transformations), in each fibre, about the point of cross-section and defines parallel transport on M_4 through the construction of the covariant derivative,

$$\bar{D}_\mu = \partial_\mu - \frac{i}{2} \bar{B}_\mu{}^{ab} M_{ab} .$$

6) Conclusions

The Lorentz gauge theory which we presented in section 4 was identified as a theory of gravitation yet possessed the unsatisfactory feature that the vierbein fields did not emerge from a gauge principle. It is only in a gauge theory of $SO(2,3)$ (or $SO(1,4)$) spontaneously broken down to $SO(1,3)$ that we may account for the vierbein fields and the role they must fulfil in a theory in which gravitation is determined entirely by the geometry of space-time. The theory of section 4 may now be identified as the $SO(2,3)$ gauge theory when restricted to the unitary gauge where the Goldstone fields are set equal to zero and the non-linearly realized coset symmetry is consequently lost.

The curvature and torsion tensor fields may be defined in the usual manner (see the appendix) for this gauge theory, however for completeness we also mention that Stelle and West[69] obtained an additional geometrical interpretation of curvature and torsion through the notion of development[48] which Cartan bundles possess. Development is a unique identification of a curve, $C(x)$ on M_4, starting at x, with a curve $C^*(x)$ on the fibre of E over x, (see the appendix for more details).

Finally let us return to the Poincaré group ISO(1,3) which is related to SO(2,3) by the group contraction process,

$$SO(2,3) \xrightarrow{R \to \infty} ISO(1,3) \quad .$$

Here, R is the 'radius' of the anti de Sitter space, $\frac{SO(2,3)}{SO(1,3)}$ defined by the surface $y^A y_A = R^2$ embedded in the 5-dimensional pseudo Euclidean space with metric $\eta_{AB} = \text{diag}(+1, -1, -1, -1, +1)$. After group contraction the algebra, (61), of SO(2,3) becomes the algebra, (53), of ISO(1,3), (where $P_a = \frac{1}{R} M_{5a}$ - see section III.2 for more details).

The problem with the Poincaré group is that the theory of the non-linear realization of groups on coset spaces depends on the Lie algebra structure being such that the coset generators close on the subgroup generators, (Appendix D). This is the case for SO(2,3) but for ISO(1,3) the coset generators, $\{P_a\}$, commute. With this ISO(1,3) algebra, the realization (64) of ISO(1,3) on $\frac{ISO(1,3)}{SO(1,3)}$ is readily evaluated to give,

$$y'^a(g, y^a) = y_o{}^a + \Lambda^a{}_b\, y^b$$

and

$$h_1(g, y^a) = h$$

where

$$g = e^{iy_o^a P_a} h \quad \text{and} \quad h\, y^a P_a\, h^{-1} \equiv \Lambda^a{}_b\, y^b\, P_a .$$

It is clear that, since $h_1 = h$, the non-linear fields for the ISO(1,3) theory only transform under the subgroup SO(1,3) and are invariant under the coset transformations $g = e^{iy^a P_a}$.

Hence, although we may obtain a Poincaré gauge theory as a special (group contraction) limit of the SO(2,3) theory, the coset symmetry is entirely lost in the process. In later chapters we shall therefore be interested in gauge theories of supergravity which contain the spontaneously broken SO(2,3) element as a sub-theory.

APPENDIX TO CHAPTER 1

FIBRE BUNDLES AND MODERN DIFFERENTIAL GEOMETRY

The purpose of this appendix is to provide the necessary background in the language and concepts of modern differential geometry to appreciate the group theoretical notion of parallel transport across space-time. We shall accept without rigour the concept of a differentiable manifold as a smooth topological space, inherent in the definition of which is the ability to cover it with coordinate patches which transform differentiably into one another. A function f on a manifold M is simply a mapping, $f: M \to R$, such that for each point $x \in M$, $f(x)$ is a real number. Our interest is restricted to the class, $\mathfrak{F}(M)$, of all differentiable functions on M. Coordinates x^μ ($\mu = 1,2,\ldots, n$) on an n-dimensional manifold M_n are a set of n functions associating each point $x \in M_n$ with n real numbers (coordinates of x).

In the first part of this appendix we run through the standard definitions of the objects required to define an affine geometry on M and in the second part redefine connections on the context of fibre bundles.

Tangent Vectors and Tangent Spaces to M[41],(48)

Formal definition: A tangent vector $\bar{x}$ to M at a point x is a mapping, $\bar{x}$: $\mathfrak{F}(M_n) \to R$, Satisfying

$$\bar{x}(af_1 + bf_2) = a\,\bar{x}(f_1) + b\,\bar{x}(f_2)$$

and

$$\bar{x}(f_1 f_2) = \bar{x}\,(f_1)f_2 + f_1\bar{x}(f_2)$$

where $f_1, f_2 \in \mathfrak{F}(M)$ and $a, b, \in \mathbb{R}$.

The set of all tangent vectors to M at x is a vector space (over $\mathbb{R}$) the <u>tangent space</u> $T_x(M)$ to M at x, (the point x is identified as the zero vector in $T_x(M)$). A natural basis for this vector space is the <u>coordinate basis</u> which is defined as follows:

Choose coordinates x^μ such that $x^\mu(x) = 0$, then sufficiently near to x any $f \in \mathfrak{F}(M)$ may be written

$$f(x^\mu) = f(0) + x^\mu \left.\frac{\partial f}{\partial x^\mu}\right]_{x^\mu = 0} \qquad \text{(Taylor expansion)}$$

then using the definition of $\bar{x}$,

$$\bar{x}(f) = \bar{x}(x^\mu) \left.\frac{\partial f}{\partial x^\mu}\right]_0 .$$

Hence we identify $\bar{x} = \bar{x}(x^\mu) \left.\frac{\partial}{\partial x^\mu}\right]_x$ as the tangent vector with components $(\bar{x}(x^1), \bar{x}(x^2), \ldots, \bar{x}(x^n)) \in \mathbb{R}^n$ with respect to the coordinate basis, $\left.\frac{\partial}{\partial x^\mu}\right]_x$.

Vectors are therefore defined as directional derivative operators which act on functions to give the rate of change of the function in a specific direction. An equivalent and more intuitive definition of a tangent vector is made possible by the consideration of curves on M.

A curve c on M is a mapping $c: I \to M$, where I is the set of real numbers $[0, 1]$, such that for each $t \in I$, $c(t) = x \in M$. A vector $\bar{x}$ tangent to c at $x = c(t)$ is then defined by,

$$\bar{x} = \left.\frac{d}{dt} \text{(along } c)\right]_t .$$

Then for any function $f \in \mathfrak{F}(M)$

$$\bar{x}(f) = \frac{d}{dt} f \cdot c(t)\Big]_t \quad \varepsilon \ \mathbb{R} .$$

This definition of a tangent vector using a specific curve is particularly useful for defining the differential of a mapping between two manifolds. First we must introduce the notion of the <u>tangent bundle $T(M)$ to M</u> which is simply the union of all tangent spaces $T_x(M)$ for all $x \varepsilon M$. Hence, $T(M) \equiv \bigcup_x T_x(M)$ is a manifold (also differentiable) with twice the number of dimensions as M. Now consider a mapping Φ between two manifolds M and M' such that for each $x \varepsilon M$, $\Phi(x) = x' \varepsilon M'$, then this mapping induces a differential mapping, $\delta\Phi : T(M) \to T(M')$, between the tangent bundles in the following way:

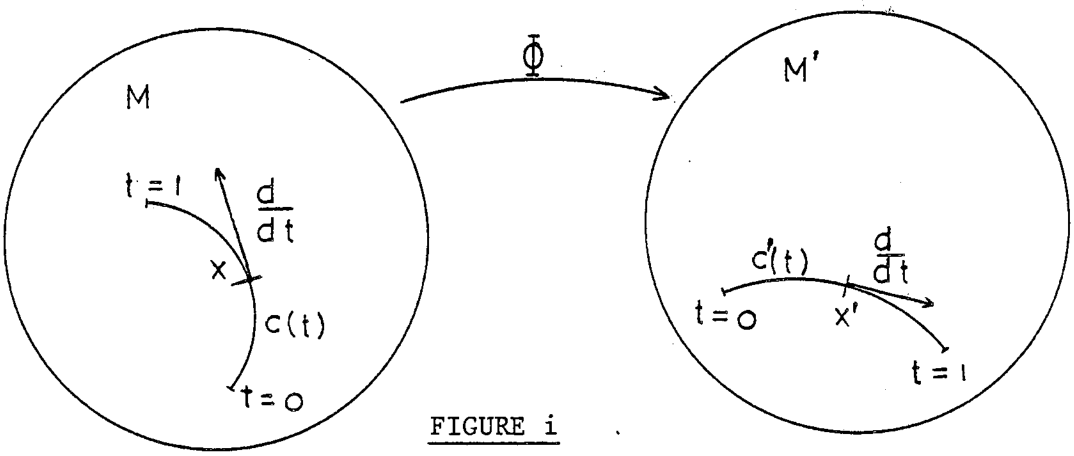


FIGURE i

In the diagram we show how Φ maps a curve $c(t)$ in M to an image curve $\Phi(c(t)) \equiv c'(t)$ in M'. It is clear from the diagram how Φ induces a mapping of the vector tangent to $c(t)$ at x into the vector tangent to $c'(t)$ at $x' = \Phi(x)$ and this is the differential mapping,

$$\delta\Phi\left(\frac{d}{dt} \text{ along } c(t)\Big]_t\right) = \frac{d}{dt} \text{ along } \Phi(c(t))\Big]_t .$$

We shall use this concept of the differential mapping on several occasions in the second section.

The mapping $P : T(M) \to M$ such that $P(T_x(M)) = x$ for $x \in M$ is called the projection of $T(M)$ onto M and is one example of the mapping Φ between manifolds.

An infinitesimal transformation, X of M is a mapping $X : M \to T(M)$ such that $P \circ X$ is the identity mapping, $x \to x$. Hence X identifies a vector $\bar{x}$ tangent to M at each point x, it may be called a cross-section on $T(M)$ or a vector field on M. If $f \in \mathfrak{F}(M)$ then Xf is also $\in \mathfrak{F}(M)$ and $Xf(x) \equiv X(x)f \equiv \bar{x}f$ for any $x \in M$. In a coordinate basis then $X(x)$ may be written as, $X(x) = A^{\mu}(x)\partial_{\mu}$ where $A^{\mu}(x)$ are the coordinate based components of the vector field. From a coordinate basis we may transform to a more general basis e_i $(i = 1, \ldots, n)$ using local general linear group transformations, $e_i(x) = G(x)_i{}^{\mu}\partial_{\mu}$ where $G_i{}^{\mu}$ are matrix elements in the fundamental $n \times n$ representation of $GL(n, R)$. Although coordinate bases vectors necessarily commute, $[\partial_{\mu}, \partial_{\nu}] = 0$ it is evident that a general basis $e_i(x)$ will not. In fact

$$\underline{[e_i, e_j] = c_{ij}{}^{k} e_k} \tag{1}$$

where

$$C_{ij}{}^{k}(x) = G_i{}^{\mu}(\partial_{\mu} G_j{}^{\nu})G_{\nu}{}^{k} - G_j{}^{\mu}(\partial_{\mu} G_i{}^{\nu})G_{\nu}{}^{k},$$

are the structure functions of the basis e_i.

For any two vector fields X and Y, the bracket $[X, Y]$ is called the Lie derivative of Y with respect to X,

$$\mathcal{L}_X(Y) = [X, Y] = Z, \text{ a new vector field.}$$

The Cotangent Space $T_x^*(M)$ and Differential Forms

Consider the mapping $\bar{\omega} : T_x(M) \to R$, thus for each vector $\bar{x}$ tangent to M, $\bar{\omega}(\bar{x})$ is a real number. The aggregate of all such $\bar{\omega}$ form an n-dimensional vector space dual to $T_x(M)$ called the cotangent space $T_x^*(M)$. The union of the cotangent spaces for each point $x \in M$ is called the cotangent bundle, $T^*(M) = \bigcup_x T_x^*(M)$. A differential 1-form $\omega(x)$ is a cross-section on $T^*(M)$ thus identifying an element $\bar{\omega}$ for each $x \in M$. It follows from the definition of $\bar{\omega}$ that ω maps vector fields into functions on M, i.e. $\omega(X) = f \in \mathfrak{F}(M)$. We may write ω in terms of a general basis θ^i $(i = 1, \ldots, n)$ as

$$\omega(x) = B_i(x)\, \theta^i(x) \quad .$$

The basis $\theta^i(x)$ is said to be dual to the vector basis $e_i(x)$ if

$$\underline{\theta^i(e_j) = \delta_j^{\ i}} \quad . \tag{2}$$

In this case,

$$\begin{aligned} \omega(x) &= B_i \theta^i (A^j e_j) \\ &= B_i A^j \theta^i(e_j) = B_i A^i \quad , \end{aligned}$$

and we see that the mapping $\omega : X \to f$ corresponds to the contraction of a covariant B_i with contravariant A^i vector in the component language.

Tensors[19]

With the tangent and cotangent bundle defined we may consider a general (p, q) type direct product bundle;

$$T(p,q) = \bigcup_x \underbrace{T_x(M) \otimes T_x(M) \otimes \ldots \otimes T_x(M)}_{p \text{ times}} \otimes \underbrace{T_x^*(M) \otimes \ldots \otimes T_x^*(M)}_{q \text{ times}}$$

A cross-section on this bundle is a (p,q) type tensor field, written in a direct product basis as,

$$t_{p,q}(x) = t^{j_1 \ldots j_p}_{i_1 i_2 \ldots i_q}(x)\, \theta^{i_1} \otimes \theta^{i_2} \otimes \ldots \otimes \theta^{i_q} \otimes e_{j_1} \otimes \ldots \otimes e_{j_p} .$$

For a specific example we consider the Riemannian metric tensor $g(x)$ which is a type (0, 2) tensor field written in a general basis as $g(x) = g_{ij}(x)\theta^i \otimes \theta^j$. A type (0,2) tensor field maps two vector fields to $\mathfrak{F}(M)$ so that for any two vector fields u, v,

$$\begin{aligned} g(u,v) &= g_{ij}\, \theta^i \otimes \theta^j (U^k e_k, V^\ell e_\ell) \\ &= g_{ij}\, U^k V^\ell \theta^i(e_k)\theta^j(e_\ell) \\ &= g_{ij}\, U^i V^j . \end{aligned}$$

$g(u,v)$ is called the inner product of the vector fields u,v and as a special case $\sqrt{g(u,u)}$ is the norm (length) of u at each point $x \in M$. Clearly a Riemannian metric is a special structure imposed on M, it is not required for the remaining definitions in this appendix.

Exterior Derivatives[(41),(56)]

A q-form is an antisymmetric tensor of type (0, q) with basis

$$\theta^{i_1} \wedge \theta^{i_2} \wedge \ldots\ldots \wedge \theta^{i_q}$$

where $\theta^1 \wedge \theta^2 \equiv \frac{1}{2}(\theta^1 \otimes \theta^2 - \theta^2 \otimes \theta^1)$, etc.

If ω is a q_1 form and σ a q_2 form then $\omega\wedge\sigma$ is a $(q_1 + q_2)$

form and

$$\omega \wedge \sigma = (-1)^{q_1 q_2} \sigma \wedge \omega \ .$$

<u>Definition</u>: the <u>exterior derivative</u> d is a mapping

$$d : q\text{-forms} \rightarrow (q+1)\text{-forms}$$

satisfying the following conditions.

i) For $f \in \mathfrak{F}(M)$ then df is a 1-form (so that f must be a 0-form) with

$$df(X) = Xf, \quad \text{for any vector field } X.$$

ii) $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^q \omega_1 \wedge d\omega_2$

(where ω_1 is a q-form).

In particular if ω_1 is a 0-form f then,

$$d(f \wedge \omega_2) \equiv d(f\omega_2) = df \wedge \omega_2 + f d\omega_2 .$$

iii) $ddf = 0$ for all $f \in \mathfrak{F}(M)$.

From condition i) we identify the components of df in a general basis as, $(df)_i = e_i f$. Consider now dx^μ, where x^μ are coordinates on M, then in a general basis

$$dx^\mu = (dx^\mu)_i \, \theta^i = (e_i \, x^\mu) \theta^i \ .$$

In a coordinate basis this becomes, $dx^\mu = \delta_\nu{}^\mu \theta^\nu = \theta^\mu$ hence <u>dx^μ is the coordinate basis</u> dual to ∂_μ.

Let ω be a 1-form written in a coordinate basis as $\omega = B_\mu \, dx^\mu$ then

$$d\omega = d B_\mu \wedge dx^\mu + B_\mu \, dd \, x^\mu$$

$$= \partial_\nu B_\mu \, dx^\nu \wedge dx^\mu + 0 \qquad \text{(using iii))}$$

$$d\omega = \tfrac{1}{2}(\partial_\mu B_\nu - \partial_\nu B_\mu) dx^\mu \wedge dx^\nu .$$

Exterior differentiation is thus a generalized curl operation on covariant tensors.

$d\omega$ is an 'exact' 2-form and quite generally satisfies[41]

$$\underline{d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \tfrac{1}{2}\omega([X,Y])} \tag{3}$$

for all vector fields X,Y.

In terms of the bases this relation yields,

$$d\theta^i(e_j,e_k) = e_j(\delta_k{}^i) - e_k(\delta_j{}^i) - \tfrac{1}{2}c_{jk}{}^i$$

so that

$$\underline{d\theta^i = -\tfrac{1}{2}\, c_{jk}{}^i\, \theta^j \wedge \theta^k} \tag{4}$$

(Recall $[e_i,e_j] = c_{ij}{}^k e_k$) .

<u>Affine Connections</u>[19], [41]

An affine connection on M may be defined as a rule which assigns to each vector field X, a mapping ∇_X of vector fields into vector fields, satisfying,

i) $\nabla_{fX + gY} = f\,\nabla_X + g\nabla_Y, \qquad f, g \in \mathfrak{F}(M)$

ii) $\nabla_X(fY) = (Xf)Y + f\,\nabla_X(Y)$

and

$$\nabla_X(f) = Xf \quad (= df(X)).$$

∇_X is often referred to as 'covariant differentiation along X'. We see from ii) that

$$\nabla_{e_i}(A^j e_j) = (e_i A^j)e_j + A^j\,\nabla_{e_i}(e_j)\ .$$

Now define $\nabla_{e_i}(e_j) = \Gamma_{ij}{}^k(x)\, e_k$

where <u>$\Gamma_{ij}{}^k(x)$ are the components of the affine connection.</u>

Then

$$\nabla_{e_i}(A^j e_j) = (e_i A^k + \Gamma_{ij}{}^k A^j) e_k$$

or, in a coordinate basis with $\nabla_{e_\mu} = \nabla_{\partial_\mu} \equiv \nabla_\mu$

$$\nabla_\mu (A^\nu \partial_\nu) = (\partial_\mu A^\nu + \Gamma_{\mu\kappa}{}^\nu A^\kappa) \partial_\nu \ .$$

The operation ∇_X may be extended to any (p,q) type tensor with the additional rule,

iii) $\nabla_X(S \otimes T) \doteq \nabla_X(S) \otimes T + S \otimes \nabla_X(T)$

for any tensors S, T.

Then, from ii) we require that

$$\nabla_X(\omega(Y)) = X(\omega(Y))$$

for any 1-form ω and vector field Y. This condition is only satisfied if

$$\nabla_{e_i}(\theta^k) = -\Gamma_{ij}{}^k \theta^j \ .$$

This rule together with $\nabla_{e_i}(e_j) = \Gamma_{ij}{}^k(e_k)$ are all that is required to form the covariant derivative of any (p,q) type tensor.

From the $\Gamma_{ij}{}^k$ we define the matrix valued <u>connection 1 form</u> $\Gamma = \theta^i \Gamma_i$, with matrix elements, $\Gamma_j{}^k = \theta^i \Gamma_{ij}{}^k$. The definition $\nabla_{e_i}(e_j) = \Gamma_{ij}{}^k e_k$ may now be written, $\nabla_{e_i}(e_j) \otimes \theta^i \equiv \nabla(e_j) = \Gamma_j{}^k \otimes e_k$ and we can define the affine connection as the mapping,

$$\nabla : (p,q) \to (p, q+1) \text{ tensor.}$$

In particular

$$\nabla(A^i e_i) = ((e_i A^k)\theta^i + \Gamma_j{}^k A^j) \otimes e_k$$

i.e. $$\underline{\nabla(A^i e_i) \;=\; (dA^k + \Gamma_j^{\cdot k} A^j) \otimes e_k} \tag{5a}$$

similarly

$$\underline{\nabla(B_i \theta^i) \;=\; (dB_k - \Gamma_k^{\cdot j} B_j) \otimes \theta^k} \tag{5b}$$

Transformation of $\Gamma_i^{\cdot j}$ under a Local Change of Basis

Consider the transformation of the basis e_i under the element $G_i^{\cdot j}$ of the general linear group,

$$e_i \rightarrow e_i' \;=\; G_i^{\cdot j} e_j, \quad \text{or in matrix notation,} \quad e' \;=\; G\,e.$$

A vector field $V = V^i e_i$ is invariant under a change of basis so that

$$V'^i \;=\; (G^{-1})^i_j V^j, \quad \text{or} \quad V'^T \;=\; V^T G^{-1}.$$

Furthermore, the 1-form basis θ^i dual to e_i must transform so that $\theta'^i(e'_j) = \delta^i_j$, hence,

$$\theta^i \rightarrow \theta'^i \;=\; (G^{-1})^i_j \theta^j \quad \text{or} \quad \theta'^T \;=\; \theta^T G^{-1}.$$

The 1-form $\omega_i = \theta^i \omega_i$ must be invariant under a change of basis so that, $\omega_i \rightarrow \omega_i' = (G)_i^{\cdot j} \omega_j$ or $\omega' = G\omega$.

The transformation of the connection form $\Gamma_j^{\cdot k} = \theta^i \Gamma_{ij}^{\cdot\cdot k}$ is determined by the requirement that $\nabla(A^i e_i)$ or $\nabla(\theta^i B_i)$ are invariant under a change of basis. The first of these implies (from 5a)) that,

$$(dA^k + \Gamma_j^{\cdot k} A^j)' \;=\; (G^{-1})^k_m (dA^m + \Gamma^m_n A^n)$$

or, in matrix notation

$$(dA + \Gamma^T A)'^T \;=\; (dA + \Gamma^T A)^T G^{-1}$$

from which we deduce,

$$\underline{\Gamma' = G\,\Gamma\,G^{-1} - G\,d\,G^{-1}} \tag{6a}$$

or

$$\underline{\Gamma'^{k}_{j} = G_{j}^{\ n}(G^{-1})^{k}_{m}\,\Gamma^{m}_{n} - G_{j}^{\ m}\,d(G^{-1})^{k}_{m}} \ . \tag{6b}$$

For later reference we point out here that the affine connection form transforms, according to (6), exactly as the gauge potentials of the group $GL(n, R)$.

Parallelism[(41),(57)]

Consider a curve $c(t)$ on M and recall that the tangent vector to $c(t)$ at t is $\dot{c}(t) \equiv \frac{d}{dt}\,(\text{along } C(t))\Big]_t$. Now for any vector field $Y(x)$ with values $Y(c(t))$ on the curve, then the vectors $Y(c(t))$ are said to be parallel (with respect to ∇) along $c(t)$ if and only if,

$$\nabla_{\dot{c}(t)}(Y)\Big]_t = 0 \quad \text{for all } t \in [0,1] \ .$$

In such a case Y is covariantly constant along $c(t)$. Curves for which the tangent vectors $\dot{c}(t)$ are themselves covariantly constant are called autoparallel curves with,

$$\nabla_{\dot{c}(t)}(\dot{c}(t)) = 0 \ .$$

To make this equation more familiar, choose a coordinate basis and write,

$$\dot{c}(t) = \frac{d}{dt}\Big]_t = \frac{dx^{\mu}(t)}{dt}\,\partial_{\mu} \ ,$$

then

$$\nabla_{\dot{c}(t)}(\dot{c}(t)) = \frac{dx^{\mu}}{dt}\left(\partial_{\mu}\frac{dx^{\kappa}}{dt} + \Gamma_{\mu\nu}^{\ \ \kappa}\,\frac{dx^{\nu}}{dt}\right)\partial_{\kappa} \ .$$

Hence $c(t)$ is an autoparallel curve if,

$$\frac{d^2x^\kappa}{dt^2} + \Gamma_{\mu\nu}{}^\kappa \frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0 .$$

On a Riemannian manifold this is identified as the geodesic equation (I.2) provided $\Gamma_{\mu\nu}{}^\kappa = \{{}^{\kappa}_{\mu\nu}\}$.

Curvature and Torsion[41]

On a manifold with an affine connection, the curvature, R, and torsion, S, tensor fields are defined by

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

and

$$S(X,Y) = \nabla_X(Y) - \nabla_Y(X) - [X,Y] .$$

To extract components recognise that R is a type (1,3) tensor which maps 3 vector fields and 1 1-form into $\mathfrak{F}(M)$, i.e. $\omega(R(X,Y)\cdot(Z)) \in \mathfrak{F}(M)$.

In terms of a basis we see that,

$$\underline{R^m_{nij} \equiv \theta^m(R(e_i,e_j)\cdot(e_n))}$$

$$= \underline{e_i\,\Gamma^m_{jn} - e_j\,\Gamma^m_{in} + \Gamma_{jn}{}^k\,\Gamma_{ik}{}^m - \Gamma_{in}{}^k\,\Gamma_{jk}{}^m - C_{ij}{}^k\,\Gamma_{kn}{}^m} \tag{7}$$

and in particular, for a coordinate basis,

$$R^\mu{}_{\nu\kappa\lambda} = \partial_\kappa\,\Gamma_{\lambda\nu}{}^\mu - \partial_\lambda\,\Gamma_{\kappa\nu}{}^\mu + \Gamma_{\lambda\nu}{}^\omega\,\Gamma_{\kappa\omega}{}^\mu - \Gamma_{\kappa\nu}{}^\omega\,\Gamma_{\lambda\omega}{}^\mu$$

which is the usual expression for the components of the Riemann curvature tensor.

Similarly for the torsion tensor, we have,

$$\underline{S_{ij}{}^k \equiv \theta^k(S(e_i,e_j)) = \Gamma_{ij}{}^k - \Gamma_{ji}{}^k - C_{ij}{}^k} \tag{8}$$

and in a coordinate basis,

$$S_{\mu\nu}{}^{\kappa} = \Gamma_{\mu\nu}{}^{\kappa} - \Gamma_{\nu\mu}{}^{\kappa}$$

which was our definition (I.12) of torsion in the main text. Curvature and torsion are also defined through the Cartan structure equations for an affine manifold.

The first of these equations reads,

$$\underline{d\theta^i + \Gamma^i_{\cdot j} \wedge \theta^j = S^i \equiv \tfrac{1}{2} S_{jk}{}^{i}\, \theta^j \wedge \theta^k} \, . \qquad (9)$$

Using (4) we easily check that (9) is equivalent to (8). The second structure equation reads,

$$\underline{d\Gamma^i_{\cdot j} + \Gamma^i_{\cdot k} \wedge \Gamma^k_{\cdot j} = R^i_{\cdot j} \equiv \tfrac{1}{2} R^i{}_{jk\ell}\, \theta^k \wedge \theta^\ell} \qquad (10)$$

and is equivalent to (7).

Lie Groups and Lie Transformation Groups[(33),(41),(48)].

Before moving on to consider fibre bundles it is useful at this stage to point out that a Lie group G is a differentiable manifold with a group multiplication structure, $G \times G \to G$, (i.e. for $g_1, g_2 \in G$ then $g_1 g_2 = g_3 \in G$). Since G is a differentiable manifold we may define vectors $\bar{g}$ tangent to G at the point g. Following our previous notation we write $\bar{g} \in T_g(G)$, the tangent space to G at g and form the tangent bundle, $T(G) = \bigcup_g T_g(G)$. A vector field X on G is a cross-section on $T(G)$ such that for each $g \in G$ then $X(g) = \bar{g} \in T_g(G)$. With the group multiplication structure we can define left and right invariant vector fields on G as follows:

A left translation L_ρ of G is simply $L_\rho : g \to \rho g$ for all $g \varepsilon G$ (ρ is any fixed element of G). This translation (or left multiplication) induces a differential mapping, (see earlier)

$\delta L_\rho : T_g(G) \to T_{\rho g}(G)$. A left invariant vector field X on G is one for which, $\delta L_\rho X(g) = X(\rho g)$, so that the vector field at ρg is the value of the field left translated from g, for all $g \varepsilon G$.

Now given any tangent vector $\bar{e} \varepsilon T_e(G)$, where e is the identity element of G, then there exists a unique left invariant vector field, X, such that $X(e) = \bar{e}$. Hence with $\bar{e}_1, \bar{e}_2 \varepsilon T_e(G)$ and $\bar{e}_1 = X(e)$, $\bar{e}_2 = Y(e)$ then $[X,Y]$ is a left invariant vector field on G and $[X,Y](e)$ is a <u>Lie algebra</u> in the vector space $T_e(G)$ induced by the <u>Lie bracket</u> of the left invariant vector fields X and Y. Choosing a basis X_i in $T_e(G)$ the Lie algebra is written,

$[X_i, X_j] = f_{ij}{}^k X_k$, where $f_{ij}{}^k$ are the structure constants of the Lie algebra $\mathfrak{g}$ of G which we now identify with $T_e(G)$. Any tangent vector $\bar{g} \varepsilon T_g(G)$ may be left (or right) translated by g^{-1} into $T_e(G)$, i.e. $g^{-1}\bar{g} \varepsilon T_e(G) \cong \mathfrak{g}$.

<u>A Lie group of transformations</u> is a Lie group G, a differentiable manifold M and a mapping,

$\phi : G \times M \to M$ such that $\phi(g,x) = x' \varepsilon M$ for any $g \varepsilon G$ and $x \varepsilon M$. For brevity the action of G on M is denoted by left <u>or</u> right (<u>not</u> both) translations, for example, for left translations, $x' = gx \equiv L_g x$. If M is a vector space and the action of G on M is linear then M forms a representation space for G. More generally the action of G on M (inducing coordinate transformations in M) gives a non-linear realization of G. A vector field X on M is left invariant if $\delta L_g X(x) = X(gx)$ where δL_g is the differential of the mapping $\phi: G \times M \to M$. Finally we mention that if $\bar{e} \in T_e(G) \cong \mathfrak{G}$, then $\bar{e}x \in T_x(M)$ for all $x \in M$ and the transformation,

$$x \to \bar{x} = \bar{e}x \text{ (for all } x) \text{ is an infinitesimal}$$

transformation of M by $\mathfrak{G}$.

FIBRE BUNDLES

Definition of a Principal Fibre Bundle, $P(M,G)$

A principal fibre bundle is a manifold, P, (for all our purposes, a differentiable manifold), together with the following collection of objects and conditions:

1) Each P has a Lie group G which acts as a Lie transformation group on P. Hence for $u \in P$ and $g \in G$ then with G acting on the right (by convention) $ug = u' \in P$. The transformation law is associative so that, $(u\, g_1)g_2 = u(g_1 g_2)$ for all $g_1, g_2 \in G$.

2) There exists a projection, $\pi : P \to M$, where M is a sub- manifold of P, such that for $u_1, u_2 \in P$ then $\pi(u_1) = \pi(u_2)$ iff $u_1 = u_2 g$ for some $g \in G$. $\pi^{-1}(x)$ is the set of all points projected onto $x \in M$ by π and is called the <u>fibre over x</u>, G_x. By definition of π it follows that G_x is isomorphic to G. M is called the <u>base manifold</u> of P and G is the <u>structure group</u> of P.

3) M has a covering of open sets U_i such that there exists a

differentiable mapping ϕ_{U_i} of $U_i \otimes G$ onto $\pi^{-1}(U_i) \equiv P|U_i$, (the part of P over U_i), and $\phi_{U_i}(x,g_1)g_2 = \phi_{U_i}(x,g_1g_2)$ for $x \in U_i$. If ϕ_M (i.e. the mapping for the whole of M) can be defined then P may be identified with $M \otimes G$ and is said to be a topologically trivial fibre bundle, $P \cong M \otimes G$.

Condition 3) requires that every fibre bundle is locally (i.e. over each U_i) trivial.

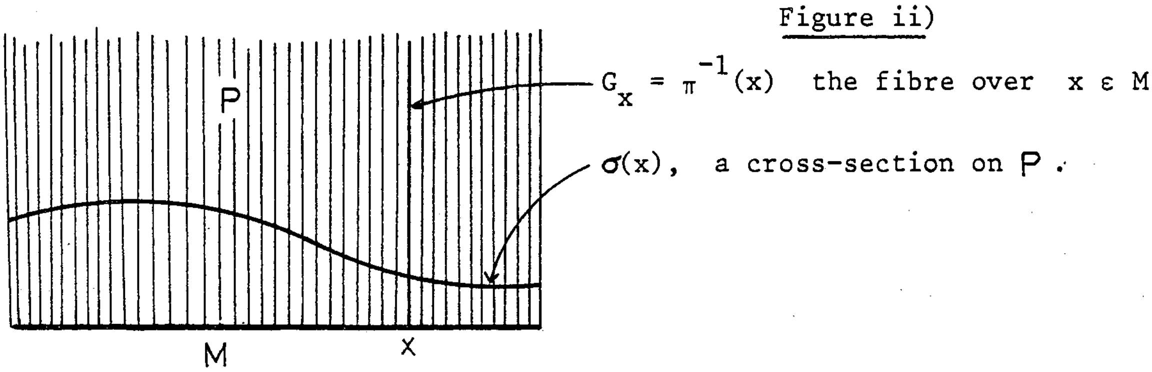
A principle fibre bundle P is denoted, $P(M,\pi, G, \{U_i\},\{\phi_{U_i}\})$ or, for brevity $P(M,G)$. Set theoretically P is simply, $P \equiv \bigcup_x G_x$, the union of the fibres G_x over each point $x \in M$. The structure group G is the standard fibre of P, copies of which form each fibre G_x over x.

We may define a cross-section, σ on P as a differentiable mapping of M into P such that

$$\pi \circ \sigma(x) = x \quad \text{for all } x \in M.$$

P is shown schematically in Figure ii) below.

Figure ii)



A gauge transformation on P is a change in the choice of cross-section $\sigma(x)$,

$$\sigma(x) \rightarrow \sigma'(x) = \sigma(x)\, g(x) \qquad \text{for all } x \in M.$$

The choice $\sigma(x)$ is a local gauge choice and $g(x)$ effects a local gauge transformation.

Fibre bundles describe physical gauge theories when $\sigma(x)$ is identified with the space-time manifold in a specified gauge. In order to describe spinor, vector, Goldstone etc. fields in this language, the associated fibre bundle must be defined. A gauge covariant procedure for identifying tangent vectors to the bundle then leads to the notion of connections on P (gauge potentials) and the curvature of P (gauge field strengths).

The Bundle E(M, G, F, P) Associated with P.

Given a principal fibre bundle $P(M,G)$ and a manifold F on which G acts effectively (by convention, on the left), then we may define an associated fibre bundle $E(M, G, F, P)$. Locally, within each open set U_i of M, (used to define P), E is simply the direct product space, $E|U_i \cong U_i \otimes F$. More generally E is the coset space, $E = \frac{P \otimes F}{G}$. The projection $\pi_E : E \to M$ is induced by the mapping $(u,\nu) \to \pi(u)$ where $u \in P$ and $\nu \in F$. More explicitly, $\pi_E \circ \rho_F(u, \nu) = \pi\circ\mu\ (u,\nu)$ where μ is the projection $P \otimes F \to P$ such that $\mu(u,\nu) = u$, and ρ_F is the natural mapping, $\rho_F : P \otimes F \to \frac{P \otimes F}{G}$ (i.e. ρ_F identifies all $u \in P$ differing only by the action of G).

Finally, the mapping,

$$\phi_{E,U_i} : U_i \otimes F \to \pi_E^{-1}(U_i) \equiv E|U_i \qquad \text{is defined,}$$

from ϕ_{U_i} of P, by

$$\phi_{E,U_i}(x, \nu) = \rho_F(\phi_{U_i}(x,e), \nu)$$

where $x \in U_i$, $\nu \in F$ and e is the identity element of G. $\pi_E^{-1}(x) \equiv F_x$

is called the fibre of E over x and F is the standard fibre. Set theoretically $E \equiv \bigcup_x F_x$. The definition of E appears extremely formal so we shall give two examples to clarify the situation.

Example 1 The tangent bundle $T(M)$ to the manifold M, with $T(M) \equiv \bigcup_x T_x(M)$, is a fibre bundle associated to the principal fibre bundle over M with structure group $GL(n,R)$ which acts effectively (and linearly) on the standard fibre, an n-dimensional linear vector space V_n, isomorphic to each fibre $T_x(M)$ over $x \varepsilon M$.

Example 2 The spinor bundle over Minkowski space-time with standard fibre $\mathbb{C}^2$, the complex space in which Weyl two component spinors lie (see Appendix A) and upon which $SL(2,c)$ acts as a transformation group. The spinor bundle is associated with the principal fibre bundle over Minkowski space-time with structure group $SL(2,c)$, often referred to as the 'bundle of Lorentz frames' (since $SO(1,3) \cong SL(2,c)$). A Weyl spinor field is simply a cross-section on the spinor bundle.

The Bundle, T(P), Tangent to P.

In physical gauge theories the central issue is the formation of a gauge-covariant derivative, D_μ of some field $\psi(x)$ (which is a cross-section on the appropriate associated fibre bundle). In the fibre bundle language the problem of forming a covariant derivative translates into one of defining a class of vectors ('horizontal vectors') tangent to $P(M,G)$. Recall that tangent vectors are derivative operators and we shall find that the covariant derivatives of gauge theories are horizontal vectors tangent to P at each point on some cross-section $\sigma(x)$ on P.

Since P is a differentiable manifold we may consider vectors $\bar{u}$ tangent to P at any point $u \in P$, then the aggregate of all such vectors at u form the tangent space $T_u(P)$ to P at $u \in P$. The space $T(P) \equiv \bigcup_u T_u(P)$ is the tangent bundle to P. It is, in fact, a principal fibre bundle with base space $T(M)$ and structure group $T(G)$. The projection $T(P) \to T(M)$ is the differential, $\delta\pi$ of π.

<u>The Infinitesimal Connection in P</u>

Consider the projection, $\mu : T(P) \to \frac{T(P)}{G}$ which simply factors G out of the tangent bundle.

$\frac{T(P)}{G}$ is a fibre bundle associated with $T(P)$ and with standard fibre $\frac{T(G)}{G} \cong T_e(G) \cong \mathfrak{g}$, the Lie algebra of G. The projection $\delta\pi^* : \frac{T(P)}{G} \to T(M)$ is simply defined by $\delta\pi^* \circ \mu = \delta\pi$.

$\underline{\frac{T(P)}{G}}$ <u>is called the bundle of connections</u> and the cross-section,

$\Gamma : T(M) \to \frac{T(P)}{G}$ such that $\delta\pi^* \circ \Gamma(\bar{x}) = \bar{x}$ for all $\bar{x} \in T(M)$ is called an <u>infinitesimal connection in P</u>.

We see that the role of this cross-section, Γ is to pick out a subspace (of the dimension of M) of $T_u(P)$ at each point $u \in P$. This space is known as the <u>horizontal subspace of $T_u(P)$</u> and will have elements denoted $\bar{u}_H$. Since G was factored out of $T(P)$ for the definition of Γ it follows that horizontal vectors are right translation invariant under G, i.e. $(\overline{ug})_H = (\bar{u})_H g$.

Given $P(M,G)$ with a connection Γ then the notion of <u>infinitesimal parallel displacement in P</u> is defined as follows:

With $\bar{x} \in T_x(M)$ then $\bar{x}$ is said to be infinitesimally near to x (recall infinitesimal transformations of M in the first part). Now consider any u such that $\pi(u) = x$ then there is a unique $\bar{u}_H$

such that,

$$\mu(\bar{u}_H) = \Gamma(\bar{x}) \ .$$

$\bar{u}_H$ is the horizontal lift of $\bar{x}$ (with respect to Γ). The point $\bar{u}_H$ is called the point obtained by the parallel displacement of u from x to $\bar{x}$.

The identification of the horizontal subspace using the cross-section Γ is difficult to work with and we now give an equivalent definition of a connection which makes closer contact with the usual gauge theory language.

Let $\bar{u} \in T_u(P)$ and $\bar{x} \in T_x(M)$, such that $\delta\pi(\bar{u}) = \bar{x}$. The horizontal part $\bar{u}_H$ of $\bar{u}$ is by definition the point in $T_u(P)$ obtained by the parallel displacement of u from x to $\bar{x}$ so that,

$$\begin{aligned} \delta\pi(\bar{u}_H) &= \delta\pi \circ \mu^{-1} \circ \Gamma(\bar{x}) \qquad (\mu : T(P) \to \tfrac{T(P)}{G}) \\ &= \delta\pi^* \circ \Gamma(\bar{x}) \quad (\text{since } \delta\pi^* \circ \mu = \delta\pi) \\ &= \bar{x} \ , \quad (\text{by definition of } \Gamma). \end{aligned}$$

Hence $\delta\pi(\bar{u} - \bar{u}_H) = \bar{x} - \bar{x} = x$, the zero vector in $T_x(M)$. By definition of $\delta\pi$ this implies that $\bar{u} - \bar{u}_H$ is tangent to the fibre over x so we call $\bar{u}_V \equiv \bar{u} - \bar{u}_H$ the vertical component of $\bar{u}$. What has been shown therefore is that Γ leads to a unique decomposition of $T_u(P)$ into the direct sum of a horizontal and a vertical subspace such that for any $\bar{u} \in T_u(P)$, $\bar{u} \xrightarrow{\Gamma} \bar{u} = \bar{u}_H \oplus \bar{u}_V$. Clearly we may equally well define a connection as identifying the vertical or horizontal subspace of $T_u(P)$ for all $u \in P$. Now since $T(G)$ is transitive on the fibre over $\bar{x}$ it follows that there exists a unique element $\omega(\bar{u}) \in T(G)$ which translates u to $\bar{u}_V$ by right multiplication,

$$\text{i.e.} \quad \underline{u.\omega(\bar{u}) = \bar{u}_V}, \quad (\omega(\bar{u}) \in T(G)). \tag{11}$$

In fact, since $\bar{u}_V$ is tangent to $T_u(P)$ at u it is clear that $\omega(\bar{u}) \in T_e(G) \cong \mathfrak{g}$ (and not $T_g(G)$ for $g \neq e$). Hence we see that the connection may be defined by a $\mathfrak{g}$-valued differential 1-form, ω on P.

For any $u \in P$ and $\bar{g} \in T(g)$ it is clear that $u\bar{g}$ is tangent to the fibre over $x = \pi(u)$ at ug so that,

$$(u\bar{g})_V = u\bar{g}, \qquad \text{then from (11) we see that}$$

$$ug \cdot \omega(u\bar{g}) = u\bar{g} \qquad \text{for all u.}$$

Hence

$$\omega(u\bar{g}) = g^{-1}\bar{g} \tag{12a}$$

Also we know that $(\bar{u}g)_H = \bar{u}_H g$ (by definition of Γ)

$$\therefore \quad (\bar{u}g)_V = \bar{u}g - (\bar{u}g)_H = \bar{u}_V g$$

$$\therefore \quad ug \cdot \omega(\bar{u}g) = (\bar{u}g)_V = \bar{u}_V g \quad .$$

Hence,

$$\omega(\bar{u}g) = g^{-1}\,\omega(\bar{u})g \tag{12b}$$

Combining (12a) and (12b) (and using $\overline{ug} = \bar{u}g + u\bar{g}$)

$$\omega(\overline{ug}) = g^{-1}\,\omega(\bar{u})g + g^{-1}\bar{g} \quad . \tag{13}$$

This equation has the form of the inhomogeneous gauge transformations of gauge potentials in Yang-Mills theories. To identify the gauge fields we must revert to the component language by choosing bases for the vectors and 1-forms.

Description of a Connection In Terms of a Local Basis

In this section we restrict our attention to $P|U_i$, the part of P over an open coordinate patch U_i. Recall that $P|U_i$ is identified with $U_i \otimes G$ through

$$\phi_{U_i} : U_i \otimes G \rightarrow P|U_i, \qquad \text{so that}$$

$$\phi_{U_i}(x,g) = u \quad \text{for} \quad x \in U_i, \quad g \in G, \quad u \in P, \quad \pi(u) = x.$$

Let x^μ be coordinates on U_i defining a coordinate basis $\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$ for vector fields on U_i.

Consider an arbitrary cross-section, $\sigma : U_i \rightarrow P|U_i$, inducing the differential cross-section, $\delta\sigma : T(U_i) \rightarrow T(P)|T(U_i)$ so that $\delta\sigma(\bar{x}) = \bar{u} \in T_u(P)$ where $u = \sigma(x)$. We may now define the σ-dependent $\mathcal{G}$-valued potentials,

$$\underline{A_\mu(\sigma) \equiv \omega(\delta\sigma(\partial_\mu))} \quad . \tag{14}$$

Under a gauge transformation, $\sigma(x) \rightarrow \sigma'(x) = \sigma(x)\, g(x)$ inducing

$$\delta\sigma(\bar{x}) \rightarrow \delta\sigma(\bar{x})\delta g(\bar{x}), \qquad \text{where } \delta g(x) \text{ is defined as follows:}$$

The local group element $g(x)$ is essentially a mapping, $g : U_i \rightarrow G$ so that $g(x) \in G$ for all $x \in U_i$. This induces the differential mapping,

$$\delta g : T(U_i) \rightarrow T\,(G) \qquad \text{so that} \quad \delta g(\bar{x}) \in T_g(G) \quad .$$

We are interested in $\delta g(\partial_\mu)$ and refer to Fig. iii) below (which should be compared with Fig. i)).

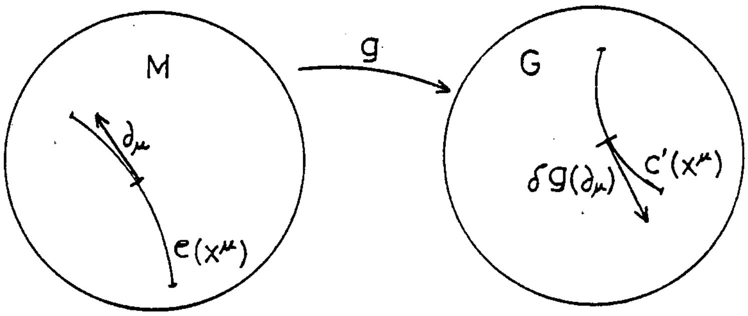


Fig. iii

$c(x^\mu)$ is the curve $\chi^\nu = \text{cons't} \quad (\nu \neq \mu)$. Hence the tangent to $c(x^\mu)$ is ∂_μ.

$$c'(x^\mu) \text{ is the image curve } = g(c(x^\mu)),$$

hence $\delta g(\partial_\mu) = \frac{d}{dx^\mu} g(c(x^\mu)) = \partial_\mu g(x).$

We now deduce the transformation of $A_\mu(\sigma)$ by using (13) with $\bar{u} \to \delta\sigma(\partial_\mu)$ and $\bar{g} \to \delta g(\partial_\mu) = \partial_\mu g(x)$,

$$\underline{A_\mu(\sigma') = g^{-1} A_\mu(\sigma) g + g^{-1} \partial_\mu g} \qquad (15)$$

and clearly the $A_\mu(\sigma)$ transform as the gauge (cross-section) dependent potentials in Yang-Mills theories. The identification mapping $\phi_{U_i} : U_i \otimes G \to P|U_i$ allows the definition of a trivial cross-section σ_g on P,

$$\sigma_g(x) = (x, g) \quad \text{for all } x, \text{ with } g \text{ fixed.}$$

We may in this case define a direct product basis ξ_μ, ξ_i for vectors $\bar{u} \in T_u(P)$ with,

$$\xi_\mu = \delta\sigma_g(\partial_\mu), \text{ the inclusion of } \partial_\mu \text{ at all points on } G_x.$$

and

ξ_i tangent to the fibre G_x over x (which is isomorphic to G).

It follows from this definition that the direct product basis satisfies the commutation rules,

$$\underline{[\xi_\mu, \xi_\nu] = [\xi_\mu, \xi_i] = 0} \text{ and } \underline{[\xi_i, \xi_j] = f_{ij}{}^k \xi_k} \qquad (16)$$

where $f_{ij}{}^k$ are the structure constants for the Lie algebra $\mathfrak{g}$ of G.

The covariant derivative D_μ in gauge theories is simply a basis for the horizontal subspace of $T_u(P)$:

Let D_μ be the horizontal lift of ∂_μ, so that

$$\mu(D_\mu) = \Gamma(\partial_\mu) ,$$

then D_μ form a basis for the horizontal subspace $\{\bar{u}_H\}$ at each point $u \in P$. It follows that D_μ, ξ_i form a basis for any vector field on P and furthermore, since $(\overline{ug})_H = \bar{u}_H g$ the basis D_μ must be translation invariant under g, so that,

$$\underline{[\xi_i, D_\mu] = 0} \quad . \tag{17}$$

Finally, notice that,

$$\begin{aligned} \delta\pi([D_\mu, D_\nu]_H) &= \delta\pi([D_\mu, D_\nu]), (\text{since } \delta\pi(\bar{u}_V) = 0) \\ &= [\partial_\mu, \partial_\nu] \qquad (\text{since } \delta\pi(D_\mu) = \partial_\mu) \\ &= 0. \end{aligned}$$

Hence,

$$\underline{[D_\mu, D_\nu] = [D_\mu, D_\nu]_V \equiv F_{\mu\nu}{}^i \xi_i} \quad . \tag{18}$$

The relationship between the direct product and the horizontal lift basis is obtained as follows:

From the definition of $\omega(\bar{u})$,

$$\begin{aligned} \omega(D_\mu) &= 0 && (\text{Since } D_\mu \text{ is horizontal}) \\ \omega(\xi_i) &= X_i && (X_i \text{ is basis for } \mathcal{G}) \\ \omega(\xi_\mu) &= \omega(\delta\sigma_g(\partial_\mu)) = A_\mu(\sigma_g) \equiv A_\mu{}^i(x,g)X_i. \end{aligned}$$

It is clear from these relations that D_μ is written in the direct product basis as,

$$D_\mu = \xi_\mu - A_\mu{}^i(x,g)\ \xi_i$$

which, up to the identification $\xi_\mu \leftrightarrow \partial_\mu$ and $\xi_i \leftrightarrow X_i$ (by isomorphisms), is the familiar form for a covariant derivative,

$$\underline{D_\mu = \partial_\mu - A_\mu{}^i(x,g)X_i} \quad . \tag{19}$$

Using (19) to calculate $[D_\mu, D_\nu]$ we find that $F_{\mu\nu}{}^i$, defined in (18), are given by

$$\underline{F_{\mu\nu}{}^i = \partial_\mu A_\nu{}^i - \partial_\nu A_\mu{}^i - f_{jk}{}^i A_\mu{}^j A_\nu{}^k} \quad . \tag{20}$$

Hence we identify $A_\mu{}^i(x,g)$ as the Yang-Mills gauge potentials and $F_{\mu\nu}{}^i(x,g)$ as the field strengths. The field strengths have the geometrical interpretation as the curvature components of P as we shall see in the next section.

The Holonomy Group and Curvature of P

Given a curve, $c(t)$ $(0 \leq t \leq 1)$ on M and a point $u_o \in P$ such that $\pi(u_o) = c(0)$, then there is a unique curve $\tilde{c}(t)$ in P such that $\pi(\tilde{c}(t)) = c(t)$ and the tangent vector to $\tilde{c}(t)$ is the horizontal lift of the tangent vector to $c(t)$ for all $0 \leq t \leq 1$. This may be written,

$$\mu\left(\frac{d}{dt} \text{ along } \tilde{c}(t)\Big|_t\right) = \Gamma\left(\frac{d}{dt} \text{ along } c(t)\Big|_t\right) .$$

Such a curve, $\tilde{c}(t)$ in P is called the horizontal lift of the curve $c(t)$ in M. The point $u_1 = \tilde{c}(1)$ at the end of this curve is called <u>the point obtained by the parallel transport of u_o along $c(t)$</u> and is briefly denoted $\underline{u_1 = c(u_o)}$. Since $(\overline{ug})_H = \overline{u}_H g$ it is clear that, if $\tilde{c}(t)$ is a horizontal curve over $c(t)$ then $\tilde{c}(t)g$ (for any $g \in G$) is also a horizontal curve over $c(t)$, hence we may write $\underline{c(u_o g)} = \underline{c(u_o)g}$.

Now consider the horizontal lift of a closed loop $c'(t)$ in M where $c'(0) = c'(1)$. It is <u>not</u> necessary that the horizontal lift of a loop is itself a loop, but the end-point $\tilde{c}(1)$ must be in the same fibre as the starting point $\tilde{c}(0)$, (i.e. $\pi(\tilde{c}(1)) = \pi(\tilde{c}(0)) = c'(0) = c'(1)$), so that $u_1 = u_o g$, for some $g \in G$. For a fixed point $u_o \in P$ and hence a fixed $x_o = \pi(u_o) \in M$ we may consider the set of all points in the same fibre, $\pi^{-1}(x_o)$, as u_o which may be reached by the parallel transport of u_o around <u>all possible loops</u> in M 'starting' at x_o. All these points may be right translated from u_o by an element g of G, hence the set of all such points in the fibre is described by a subset H of G. It is not difficult to see that this subset is in fact a <u>subgroup</u> H of G:

(1) Closure; let $g_1, g_2 \in H$ such that $u_o g_1 = c_1'(u_o)$ and $u_o g_2 = c_2'(u_o)$. Write the join of two curves $c_1(t)$ and $c_2(t)$, where $c_2(0) = c_1(1)$, as $c_2 V c_1(t)$ (which simply means go along c_1 then along c_2) and we see that

$c_2' V c_1'(u_o) = c_2'(u_o g_1) = c_2'(u_o)g_1 = u_o g_2 g_1$. Hence $\underline{g_2 g_1 \in H}$ (closure).

(2) Inverse; let $g \in H$ such that $u_o g = c'(u_o)$, then the inverse operation is to go round c' in the opposite sense (from $t = 1$ to $t = 0$) and this curve may be denoted c'^{-1}. Hence $c'^{-1}(u_o g) = c'^{-1}(u_o)g = u_o$, so that $c'^{-1}(u_o) = u_o g^{-1} \Rightarrow \underline{g^{-1} \in H}$ (inverse).

(3) Identity element simply corresponds to the trivial loop

$$c'(t) = x_o \quad \text{for all } t.$$

This subgroup $H(u_o)$ of the structure group G is called the <u>holonomy group of P with reference point u_o</u>. In fact choosing a different reference point $u_1 = u_o g$ leads to a conjugate group, $H(u_o g) = g^{-1}H(u_o)g$. Hence if M is at least pathwise connected, it makes sense to refer to the holonomy group H of P(M,G) without specifying a reference point. The holonomy group describes the path dependence of parallel transport in P with respect to a connection Γ. If the parallel transport process were path independent then loops in M would be lifted to loops in P and H would be the identity element e.

The <u>curvature R of $P(M,G)$</u>, with a connection Γ, is a local object defined at each point $u \in P$ through the consideration of the horizontal lift of <u>infinitesimal loops</u> in M. In fact an infinitesimal loop doesn't lie in M but in $T(M)$ and an infinitesimal loop at the point $x \in M$ is a finite parallelogram in $T_x(M)$ with vertices $(x, \bar{x}_1, \bar{x}_1 + \bar{x}_2, \bar{x}_2)$ where x is the zero vector in $T_x(M)$. Such a loop at x on M is therefore defined by two vectors $\bar{x}_1, \bar{x}_2 \in T_x(M)$ and will be denoted $c'(\bar{x}_1, \bar{x}_2)$. In order that parallel transport along a curve in $T(M)$ may be defined it is evident that we require a connection in the tangent bundle $T(P)(T(M), T(G))$ to $P(M,G)$. This connection, denoted $T(\Gamma)$ in the tangent bundle $T(P)$ (which is also a principal fibre bundle) is induced naturally from Γ in P as described for example in ref. (48). Here we only require two simple results:

1) Let p be the projection $p: T(P) \to P$ such that for any $\bar{u} \in T_u(P)$ then $p(\bar{u}) = u$, then for any curve $\tilde{\bar{c}}(t)$ in $T(P)$ which is horizontal with respect to $T(\Gamma)$, $p(\tilde{\bar{c}}(t))$ is horizontal in P with respect to Γ.

2) Let m be the projection $m : T(M) \to M$, then for any curve curve $\bar{c}(t)$ in $T(M)$ and any $\bar{u}_o \in T_{u_o}(P)$ such that $\delta\pi(\bar{u}_o) = \bar{c}(0)$,

$$p(\bar{c}(\bar{u}_o)) = m \circ \bar{c}(u_o) .$$

Here, $m \circ \bar{c}$ is a curve in M and $m \circ \bar{c}(u_o)$ is the point in P reached by parallel displacement with respect to Γ along $m \circ \bar{c}$. $\bar{c}(\bar{u}_o)$ is the point in $T(P)$ reached by parallel displacement along $\bar{c}$ with respect to $T(\Gamma)$. In view of the fact that the horizontal curve over $m \circ \bar{c}$, starting from u_o is unique we see that the second result follows immediately from the first.

Returning to the curve $c'(\overline{x}_1, \overline{x}_2)$ in $T_x(M)$ then for any point $u_o \in \pi^{-1}(x)$ we shall denote by $c'(\overline{x}_1, \overline{x}_2)(u_o)$ the point in $T(P)$ obtained by the parallel transport of u_o around $c'(\overline{x}_1, \overline{x}_2)$ with respect to $T(\Gamma)$. It is clear that this end point is in the same fibre $\pi^{-1}(x)$ as u_o. Furthermore since $m \circ c'(\overline{x}_1, \overline{x}_2) = x$ and hence $m \circ c'(\overline{x}_1, \overline{x}_2)(u_o) = u_o$, it follows from 'result (2)' above that $p(c'(\overline{x}_1, \overline{x}_2)(u_o)) = u_o$ which means that $c'(\overline{x}_1, \overline{x}_2)(u_o)$ is tangent to P at u_o and therefore differs from u_o by a right translation $u_o \to u_o\overline{e}$, where $\overline{e}$ $T_e(G) \cong \mathfrak{g}$. Hence the result of parallel transporting u_o around $c'(\overline{x}_1, \overline{x}_2)$ is,

$$\underline{c'(\overline{x}_1, \overline{x}_2)(u_o) = u_o \overline{e}} \qquad (\overline{e} \in \mathfrak{g})$$

$\overline{e}$ is determined entirely by $\overline{x}_1, \overline{x}_2$ and u_o, it is called <u>the curvature of P at u_o</u> and may be denoted, $R(u_o, \overline{x}_1, \overline{x}_2) \equiv \overline{e} \in \mathfrak{g}$.

The transformation of R under right translations $u_o \to u_o g$ along a fibre is easily determined.

$$c(\overline{x}_1, \overline{x}_2)(u_o g) = u_o g\, R(u_o g, \overline{x}_1, \overline{x}_2)$$

but

$$c(\overline{x}_1,\overline{x}_2)(u_o g) = c(\overline{x}_1, \overline{x}_2)(u_o)g = u_o\, R(u_o,\overline{x}_1, \overline{x}_2)g$$

hence $R(u_o g, \overline{x}_1,\overline{x}_2) = g^{-1} R(u_o g, \overline{x}_1,\overline{x}_2)g$

and we see that R transforms homogeneously in the adjoint representation of G.

From R, we may define a $\mathfrak{g}$-valued 2-form, Ω on P, the curvature 2-form,

$$\underline{\Omega(\overline{u}_1, \overline{u}_2) \equiv -\tfrac{1}{4} R(u_o, \delta\pi(\overline{u}_1), \delta\pi(\overline{u}_2))} \in \mathfrak{g}$$

where $\overline{x}_1 = \delta\pi(\overline{u}_1)$, $\overline{x}_2 = \delta\pi(\overline{u}_2)$ and $\overline{u}_1,\overline{u}_2 \in T_{u_o}(P)$.

The curvature 2-form satisfies the <u>Cartan structure equation</u>

$$d\omega + \tfrac{1}{2}[\omega, \omega] = \Omega$$

where ω is the $\mathcal{G}$ valued 1-form defining the connection Γ in P and $[\omega, \omega] \equiv [\omega^i X_i, \omega^j X_j] = \omega^i \wedge \omega^j [X_i, X_j]$.

$$= \omega^i \wedge \omega^j f_{ij}{}^k X_k \qquad (X_i \text{ is a basis for } \mathcal{G}).$$

The proof of this equation devides naturally into two parts. First notice that, since $\omega(\bar{u}_H) \equiv 0$, for horizontal vector fields on P the structure equations become,

$$d\omega(\bar{u}_{1_H}, \bar{u}_{2_H}) = \Omega(\bar{u}_1, \bar{u}_2) = -\tfrac{1}{4}R(u, \delta\pi(\bar{u}_1), \delta\pi(\bar{u}_2)).$$

This equation is verified[(1),(48)] by calculating the point in T(P) obtained by the parallel transport of u_o around $c'(\bar{x}_1, \bar{x}_2)$ and then simply using the definition of the curvature R.

The remaining part of the proof consists of verifying that for any two vector fields $\bar{u}_1, \bar{u}_2$ on P then,

$$d\omega(\bar{u}_1, \bar{u}_2) + \tfrac{1}{2}[\omega(\bar{u}_1), \omega(\bar{u}_2)] = d\omega(\bar{u}_{1_H}, \bar{u}_{2_H})$$

and this is easily done using the direct product basis, $\{D_\mu, \xi_i\}$ for vector fields on P.

The structure equation on insertion of the vectors D_μ takes the form

$$d\omega(D_\mu, D_\nu) = -\tfrac{1}{4} R(u_o, \partial_\mu, \partial_\nu) \; .$$

Using the general relation (3) for exact 2-forms and the fact that $\omega(D_\mu) = 0$ we see that the left hand side of this equation becomes $-\tfrac{1}{2}\omega([D_\mu, D_\nu])$, but we defined the Yang-Mills field strengths through the Lie bracket, $[D_\mu, D_\nu] = -F_{\mu\nu}{}^i \xi_i$, finally using $\omega(\xi_i) = X_i$ we identify,

$$F_{\mu\nu}{}^{i}\, X_{i} \;=\; \tfrac{1}{2}\, R(u_{o},\partial_{\mu},\partial_{\nu}) \quad .$$

We have therefore identified the Yang-Mills field strengths with the geometrical curvature of $P(M,G)$.

Cartan Connections (19), (20), (48), (50).

Gauge theories have a geometrical description not simply on space-time, but on an extended fibre bundle manifold with space-time identified as a cross-section on the bundle. To obtain a geometry on space-time itself we require a connection which identifies the 'horizontal vectors' with the tangent vectors to space-time, on the cross-section. Such connections are called Cartan connections and they define a soldering of the horizontal subspace to the tangent space to the base manifold at each point on a cross-section. For the special problem of geometries on space-time we refer to the main text, here we concentrate on the general properties of fibre bundles with a Cartan connection.

A Cartan connection Γ_c in a principal fibre bundle $P(M,G)$ is a cross-section,

$$\Gamma_c \;:\; T(M) \;\rightarrow\; \frac{T(P)}{G} \quad ,$$ (i.e. a connection as previously

defined), which also defines the soldering of the associated bundle $E(M, F, G, P)$ to M.

Definition: A bundle $E(M, F, G, P)$ associated with $P(M,G)$ is soldered to M if it satisfies the following conditions.

(1) G acts transitively on F, so that F is the orbit of G with respect to any arbitrary point of F. Hence F is isomorphic to the coset space $\frac{G}{H}$ (see Appendix D) where H is the subgroup of G which leaves the arbitrary point fixed. We therefore see that the standard fibre of E is $F \cong \frac{G}{H}$.

(2) $\dim M \;=\; \dim F \;\equiv\; \dim \frac{G}{H}$ (hence $\dim G > \dim M$).

(3) E admits a cross-section, which is to be identified with M. Once a cross-section is picked the structure group reduced to H.

(4) The tangent space $T_x(F_x)$ to F_x at x (i.e. on the cross-section) is identified with the tangent space $T_x(M)$ to M by an isomorphism, $\gamma : T_x(M) \to T_x(F_x)$. In particular $\gamma(x) = x$. It is this isomorphism γ which describes the soldering of E to M.

The fact that the Cartan connection Γ_c is required to define the soldering, combined with the usual role of lifting tangent vectors to M into a horizontal subspace means that all horizontal vectors in T(P) must be tangent to the coset, $\frac{G}{H}$ degrees of freedom along each fibre. Hence there must be no horizontal vectors tangent to the sub-bundle P'(M,H) of P(M,G). If <u>ω is the $\mathfrak{g}$-valued form on P defining a Cartan connection</u> then, for any $\bar{u}' \in T(P')$, ω must satisfy,

$$\omega(u'\,\bar{h}) = h^{-1}\,\bar{h} \qquad (h \in H)$$

$$\omega(\bar{u}'\,h) = h^{-1}\,\omega(\bar{u}')h$$

<u>and</u> $$\underline{\omega(\bar{u}') = 0 \Rightarrow \bar{u}' = 0.}$$

The first two conditions are required for any connection on P', it is the third condition which identifies ω as the form defining a Cartan connection since it states that there are no non-zero vectors in T(P') which are horizontal with respect to the connection.

The algebra $\mathfrak{g}$ of G decomposes into

$$\mathfrak{g} = \mathfrak{h} \oplus t$$

where $\mathfrak{h}$ is the algebra of H and t, the coset tangent space. We are interested in algebras for which this decomposition satisfies,

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$$

$$[\mathfrak{h},t] \subset t$$

and $$[t,t] \subset \mathfrak{h}$$

Using the decomposition of the algebra we write,

$$\omega = \omega' + \theta$$

where ω' is the $\mathfrak{h}$-valued form defining a connection on P' and θ is a t-valued form called the <u>form of soldering</u>. From the definition of ω, which required that $\omega(\bar{u}') = 0 \Rightarrow \bar{u}' = 0$, it follows that $\underline{\theta(\bar{u}') = 0 \Longrightarrow \bar{u}' = \bar{u}'_V}$. Also the requirement that $\omega(\bar{u}'h) = h^{-1}\omega(\bar{u}')h$ and the decomposition of $\mathfrak{g}$ imply that $\underline{\theta(\bar{u}'h) = h^{-1}(\bar{u}')h}$.

The Cartan connection on P is therefore equivalent to an 'ordinary''connection on P' together with the 'form of soldering' which identifies the horizontal vectors in $T(P)$ with the tangent vectors to M on a cross-section. Substituting $\omega = \omega' + \theta$ into the Cartan structure equation, $d\omega + \frac{1}{2}[\omega,\omega] = \Omega$ and using the decomposition of the algebra we see that it splits into an $\mathfrak{h}$-valued and a t-valued equation,

$$d\omega' + \tfrac{1}{2}[\omega', \omega'] = \Omega_{\mathfrak{h}} - \tfrac{1}{2}[\theta,\theta]$$

and

$$d\theta + [\omega', \theta] = \Omega_t .$$

We have recovered the two structure equations of Cartan where

$\Omega_{\mathfrak{h}} - \frac{1}{2}[\theta, \theta] \equiv \Omega'$ is the curvature form of the Cartan connection in P' and

Ω_t is the torsion form of the Cartan connection in P'.

To proceed further we should choose some specific group structure but leave this for the main text. We simply remark here that the choice $G = A(n, R)$ (the affine group) and $H = GL(n, R)$ yield a group theoretical definition of the affine geometry on M which we discussed earlier (see reference (50)).

In addition to the usual parallel transport process which any connection may define, there is an additional geometrical construction in bundles, P, with a Cartan connection called <u>development</u> which we shall briefly describe to conclude this appendix.

Consider a curve $c(t)$ on M and a curve $\tilde{c}(t)$ in $P'(M,H)$, covering $c(t)$, such that $\pi(\tilde{c}(t)) = c(t)$. Each point of $\tilde{c}(t)$ may be parallel displaced (with respect to the Cartan connection Γ_c) back to the fibre $\pi^{-1}(x_o)$ over $x_o = c(0)$. In this manner a curve $\tilde{\tilde{c}}(t)$ in $\pi^{-1}(x_o)$ is defined (with $\tilde{\tilde{c}}(0) = \tilde{c}(0)$) and since $\tilde{c}(t)$ lies in P' it cannot be horizontal with respect to Γ_c so that $\tilde{c}(t)$ is never trivial.

Defining the natural coset projection,

$$\rho_H \ : \ P(M,G) \ \rightarrow \ E(M,\tfrac{G}{H}, G, P) \ \equiv \ \frac{P}{H}$$

then the curve $c^*(t) = \rho_H(\tilde{\tilde{c}}(t))$ lies in the fibre $\pi_E{}^{-1}(x_o)$. The importance of $c^*(t)$ lies in the fact that it is defined independently of the choice of curve $\tilde{c}(t)$ over $c(t)$:

Choosing a different curve $\tilde{c}'(t)$ over M (but still in P') we see that $\tilde{c}'(t) = \tilde{c}(t)\,h(t)$ $(h \in H)$. Since parallel transport and right multiplication by G commute $((\overline{ug})_H = \overline{u}_H g)$ it then follows that,

$$\tilde{\tilde{c}}'(t) \ = \ \tilde{\tilde{c}}(t)\,h(t) \qquad \text{and so}$$

$$c^*(t) \ = \ \rho_H\,(\tilde{\tilde{c}}'(t)) \ = \ \rho_H(\tilde{\tilde{c}}(t)).$$

Given a curve $c(t)$ on M, the curve $c^*(t)$ in the fibre of $E(M, \frac{G}{H}, G, P)$ over $c(0)$ is <u>uniquely</u> determined by the above process and is called the development of $c(t)$ into $\pi_E{}^{-1}(C(0))$.

We remark finally that the converse of this statement is also true,[48] i.e. given a curve $c^*(t)$ in $\pi_E^{-1}(x_o)$ then there exists a unique curve in M, starting at x_o, which may be developed into $c^*(t)$.

CHAPTER II

SUPERGRAVITY THEORY - THE SUPERSYMMETRIC COUPLING OF MATTER TO GRAVITY

1) Einstein Gravity and Particle Physics

As a classical theory, gravitation is a manifestation of the geometry of space-time, determined entirely by the components $g_{\mu\nu}$ and $S_{\mu\nu}{}^{\kappa} = \Gamma_{[\mu\nu]}{}^{\kappa}$ of the metric and torsion tensor fields (respectively). The coupling of matter fields to gravitation is determined by the formation of the appropriate covariant derivatives in curved space-time, as described in sections I.2 and I.4. The matter fields may be quantized according to the well established prescriptions of modern quantum field theory and the coupling to gravity then treated semi-classically through the study of quantum field theory in curved space-time(38). However, the gravitational field itself possesses dynamical degrees of freedom in the metric tensor $g_{\mu\nu}(x)$, which is a propagating field, (not so the torsion tensor field, see equation (I.43)). The geometry of space-time is not frozen, like the flat geometry of Minkowski space-time, but propagates information about gravitational disturbances, at the speed of light, in the form of gravitational waves. For consistency with the principles of quantum physics we must therefore be able to quantize the gravitational field and describe perturbative gravitational interactions in terms of the exchange of 'gravitons', the quanta of the gravitational field.

We begin this chapter by briefly describing how the graviton field emerges in Einstein gravitation and the problems with the quantization of this theory. These problems are of a nature which motivates our interest in theories which involve a highly symmetric coupling of matter to gravity.

The dynamics of gravitation are contained in the Einstein action,

$$\underline{\mathcal{L}_E = \frac{1}{\kappa^2}\sqrt{-g}\,R = \frac{1}{\kappa^2}\sqrt{-g}\,g^{\mu\nu}R^{\lambda}{}_{\mu\nu\lambda}} \tag{1}$$

where (equation (I.8b))

$$R^{\lambda}{}_{\mu\nu\lambda} = \partial_\mu \{{}^{\lambda}_{\nu\lambda}\} - \partial_\lambda \{{}^{\lambda}_{\mu\nu}\} + \{{}^{\kappa}_{\mu\lambda}\}\{{}^{\lambda}_{\kappa\nu}\} - \{{}^{\kappa}_{\mu\nu}\}\{{}^{\lambda}_{\kappa\lambda}\}$$

and (equation (I.2))

$$\{{}^{\lambda}_{\mu\nu}\} = \frac{1}{2} g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}) .$$

Now in a coordinate system which becomes Lorentzian as the gravitational coupling vanishes $(\kappa \to 0)$ we may write the covariant metric tensor as,

$$\underline{g_{\mu\nu} = \eta_{\mu\nu} + \kappa\,\phi_{\mu\nu}} \quad . \tag{2}$$

This defines the symmetric tensor field $\phi_{\mu\nu} = \phi_{\nu\mu}$, with the dimensions of mass, $\underline{[\phi_{\mu\nu}] = \text{mass}}$.

The contravariant metric tensor $g^{\mu\nu}$ is defined by the requirement that, $g^{\mu\nu}g_{\nu\kappa} = \delta^{\mu}_{\kappa}$. Substituting (2) into this equation we obtain an iterative solution for $g^{\mu\nu}$,

$$\underline{g^{\mu\nu} = \eta^{\mu\nu} - \kappa\phi^{\mu\nu} + \kappa^2\phi^{\mu\lambda}\phi_{\lambda}{}^{\nu} + O(\kappa^3)} \quad . \tag{3}$$

The left hand side is an infinite series in $(\kappa\phi_{\mu\nu})^n$ where the indices on the $\phi_{\mu\nu}$ are contracted using the zeroth order metric $g_{\mu\nu} = \eta_{\mu\nu}$, $g^{\mu\nu} = \eta^{\mu\nu}$. In this sense $\phi_{\mu\nu}$ may be regarded as a flat space-time tensor field. Substituting (2) and (3) into $\mathcal{L}_E$ we obtain an infinite series in powers of $\phi_{\mu\nu}$,

$$\underline{\mathcal{L}_E = \mathcal{L}_{(2)} + \kappa\mathcal{L}_{(3)} + \kappa^2\mathcal{L}_{(4)} + O(\kappa^3)} \tag{4}$$

where we have dropped a linear term, $\frac{1}{\kappa}\mathcal{L}_{(1)}$ which is a total divergence.

The quadratic term in $\mathcal{L}_E$ is,

$$\mathcal{L}_{(2)} = \underline{2\partial^\nu\phi^\kappa{}_\lambda\partial_\kappa\phi^\lambda{}_\nu - \partial^\nu\phi^\kappa{}_\lambda\partial_\nu\phi_\kappa{}^\lambda - 2\partial^\nu\phi^\kappa{}_\nu\partial_\kappa\phi^\lambda{}_\lambda}$$
$$\underline{+ \partial^\nu\phi^\kappa{}_\kappa\partial_\nu\phi^\lambda{}_\lambda} \tag{5}$$

and determines the free propagator for the $\phi_{\mu\nu}$.

The terms $\kappa^{n-2}\mathcal{L}_{(n)}$ $(n = 3,4, \ldots \infty)$ are self interaction terms, (or n-point bare vertices), for the $\phi_{\mu\nu}$.

We therefore see that the Einstein action may be regarded as the action for a very complicated self interacting theory of the tensor field $\phi_{\mu\nu}$ on flat space-time. To lowest order in κ this theory becomes the free field theory $\mathcal{L}_{(2)}$ with the classical field equations,

$$\frac{\delta\int\mathcal{L}_{(2)}d^4x}{\delta\phi_{\mu\nu}} = \partial^\kappa\partial_\mu\,\phi_{\nu\kappa} + \partial^\kappa\partial_\nu\,\phi_{\mu\kappa} - \partial^\kappa\partial_\kappa\,\phi^\lambda{}_\lambda$$
$$\partial_\mu\partial_\nu\,\phi^\kappa{}_\kappa - \eta_{\mu\nu}(\partial^\kappa\partial^\lambda\,\phi_{\kappa\lambda} - \partial^\kappa\partial_\kappa\,\phi^\lambda{}_\lambda) = 0\ .$$

Contracting these equations with $\eta^{\mu\nu}$ we find that

$$\partial^\kappa\partial^\lambda\,\phi_{\kappa\lambda} - \partial^\kappa\partial_\kappa\,\phi^\lambda{}_\lambda = 0$$

hence the field equations reduce to

$$\partial^{\kappa}\partial_{\kappa}\,\phi_{\mu\nu} - \partial^{\kappa}\partial_{\mu}\,\phi_{\nu\kappa} - \partial^{\kappa}\partial_{\nu}\,\phi_{\mu\kappa} + \partial_{\mu}\partial_{\nu}\,\phi^{\kappa}_{\ \kappa} = 0 \qquad (6)$$

These are the equations for a zero rest mass, spin 2 field $\phi_{\mu\nu}$ (see for example ref. (52)). Furthermore Weinberg has shown[(73)] that interactions mediated by particles of spin 2 require that all particles of matter have the same interaction 'charge'. At low energies Weinberg identified this charge as the ratio of the particle's effective gravitational mass to its inertial mass and hence recovered a microscopic version of the equivalence principle. Since it is uniquely[(73)] spin 2 particles which lead to the principle of equivalence we shall accept that the graviton is a spin 2 massless particle (massless since gravity is an infinite range force), with the $\phi_{\mu\nu}$ as the corresponding classical field.

From the action $\mathcal{L}_E(\phi)$ of equation (4), the Feynman rules for the construction of scattering amplitudes may be written down. The quadratic part, $\mathcal{L}_{(2)}$ yields a massless free particle propagator $\sim \frac{1}{P^2}$ (in momentum space) and the interaction terms all yield bare vertices $\sim P^2$, (since there are two derivatives in each term), with complicated symmetry factors on the indices[(44)] which we have omitted. With these rules, tree diagrams and their associated physical processes have been evaluated. However, when closed graviton loops are included these simple Feynman rules become inadequate. Each loop involves an integral, $\int d^4p$, over an internal 4-momentum so that the simplest case of one loop (with N vertices) has a superficial degree of divergence, $D = 4$ and in general for n loops, $D(n) = 2n + 2$. This simple counting of powers of the momentum in the amplitude does not necessarily reflect the actual divergence of a diagram however. In Yang-Mills theories a proper use of the available gauge symmetry reduces the degree of divergence and in fact, the local symmetries of gravitation have a similar effect in the present case[44]. In one respect, counting the powers of the momentum in an amplitude is still a valuable guide. Because the gravitational coupling constant κ has the dimensions of $(\text{mass})^{-1}$ it follows that there are an infinite number of possible dimensionless counterterms $\sim\kappa^{2n}\int P^{2n}$, (with a dimensionless coupling constant, all dimensionless counterterms must be logarithmic). Hence we anticipate that if non-vanishing counterterms are required at any number of loops then this will lead, at higher orders, to more and more counterterms and a non-renormalizable theory. We therefore see that a successful theory of quantum gravity requires the infinite counterterms to vanish at each order in perturbation theory.

The one graviton loop calculation was performed by 't Hooft and Veltman[42] using the background field method[18] and dimensional regularization. In this background field method, the quantum gravitational field fluctuates about a classical solution, $g_{\mu\nu}(x)$ of the Einstein field equations (I.8a), $R_{\mu\nu} = 0$, and not simply about flat Minkowski space-time. The form of the counterterms required to render the effective action finite at one loop is

$$\Delta \mathcal{L}_E \sim \frac{\sqrt{-g}}{\varepsilon} (\alpha R^2 + \beta R^{\mu\nu} R_{\mu\nu})$$

where α and β are numerical factors and $\varepsilon = n - 4$ (n is the continuous dimension of space-time). The counterterm is therefore infinite (for $n = 4$) off-shell but for the calculation of on-shell

S-matrix elements, $g_{\mu\nu}$ satisfies the field equations, $R_{\mu\nu} = 0$ and so $\Delta \mathcal{L}_E = 0$ on-shell. Gravity is therefore finite at one loop. At two loops however there exist[21] infinite counterterms $\sim (R_{\mu\nu\kappa\lambda})^3$ which don't vanish on-shell and we may now expect more and more non-vanishing counterterms at higher orders. Hence pure gravitation breaks down as a quantum theory at 2 loops.

The fact that pure gravitation, described by $\mathcal{L}_E$, breaks down is not, by itself, a disaster for the theory. This is because the notion of a gravitational field in matter-free space is lost in any event for the quantum theory. We may consider a region of space-time where there are no on-shell electrons, quarks etc., but these states will still contribute to a quantum process through, for example, vacuum polarization[4] effects such as that shown in Fig. 1.

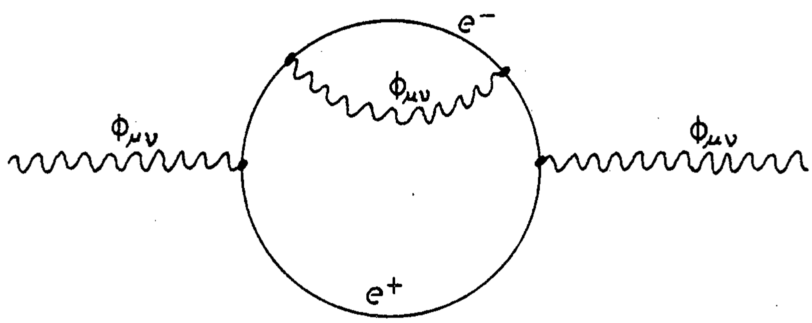


Figure 1

Bearing in mind Weinberg's statement of the equivalence principle that gravitons couple with equal 'charge' to all particles (including themselves) we see that a proper quantum theory of gravitation must, in addition to the Einstein action, include actions for the coupling of gravity to all the fundamental matter fields. In Chapter 1 we showed how all matter fields may be coupled to gravity, essentially by the replacement of ordinary derivatives and the Minkowski metric $\eta_{\mu\nu}$ by the appropriate covariant derivatives (see section I.4) and a more general metric $g_{\mu\nu}(x)$, respectively. As a simple example, the action for the coupling of gravity to a massless scalar field ϕ is,

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\, \partial_\mu\phi\partial_\nu\phi \quad .$$

't Hooft and Veltman calculated the one loop counterterms for this action using the background field method and found that[(42),(34)]

$$\Delta\mathcal{L} = \frac{1}{\varepsilon}\frac{203}{80}\sqrt{-g}\, R^2 \quad .$$

This term does not vanish on-shell however since the classical background field equations are now, $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu}$ and contract to $R = -\kappa^2 T^\mu{}_\mu \neq 0$, where $T_{\mu\nu}$ is the symmetric stress-energy tensor, calculated from $\mathcal{L}$. The non-vanishing of the one loop counterterms is not by any means just a problem for the scalar field action, rather a general feature of actions for gravity coupled to matter by the minimal coupling prescription (see for example ref. (42)).

Clearly this approach to quantum gravity is inadequate; we cannot search through the one loop counterterms for all possible matter actions expecting miraculous cancellations between them. We (the physics community) are not prepared however to abandon Einstein's theory of gravitation or modern quantum field theory (since we have nothing feasible with which to replace them) and so the most conservative step must be to impose extra symmetry requirements upon the theory which will restrict much further the form of the actions for gravity coupled to matter. To have any chance of effecting systematic cancellations between gravity and matter loop divergences this symmetry must connect the spin 2 graviton with the spin 0, $\frac{1}{2}$, 1, $\frac{3}{2}$ matter fields and hence, must involve Fermionic (half-integer spin) generators.

Such symmetries on flat space-time are known as supersymmetries[(76),(24)] and their associated supersymmetric field theories originally generated much interest precisely because of the remarkable cancellations between Fermion and Boson loops within these theories. Supergravity theories[(30), (16), (52)] are supersymmetric theories involving the spin 2 graviton, (they also necessarily involve local supersymmetry on curved space-time, see section 3), the smallest such theory, minimal supergravity involves the graviton and a spin $\frac{3}{2}$ state known as the 'gravitino'. An encouraging feature of this minimal supergravity theory is that it is finite to two loops[(21),(52)]. At three loops invariant counterterms exist but their coefficients have not yet been calculated; they may vanish. Minimal supergravity cannot be a full quantum gravity theory however since it doesn't involve any known matter states. The larger, extended supergravity theories involve states down to spin 0 and indeed the largest $(N = 8)$ theory contains sufficient states for a full quantum gravity theory and a large enough gauged internal symmetry group ('hidden SU(8)'[(52)]), to accommodate the grand unified theories of the electro-weak and strong interactions. This theory is immensely complicated, especially when the problem of symmetry breaking to that at the energy scale of the present day universe is introduced, and is apparently beyond the scope of any work!

For the remainder of this chapter we shall review supersymmetry and supergravity in preparation for the later chapters on the gauge theory structure of supergravity.

2) Supersymmetry

Linear transformations connecting Bose fields $A(x)$ and Fermi fields $\psi_\alpha(x)$, where $A(x)$ has the canonical dimensions of mass and $\psi_\alpha(x)$ has the canonical dimensions of $(\text{mass})^{3/2}$, will have the general form,

$$\underline{\delta\, A(x) \sim \varepsilon^\alpha\, \psi_\alpha} \tag{7a}$$

$$\underline{\delta\, \psi_\alpha(x) \sim \not{\partial} A(x)\,\varepsilon_\alpha} \quad . \tag{7b}$$

Here, $\varepsilon_\alpha \neq \varepsilon_\alpha(x)$ are the constant parameters of the global transformations and since the ψ_α are Grassmann variables, $\{\psi_\alpha, \psi_\beta\} \equiv \psi_\alpha\psi_\beta + \psi_\beta\psi_\alpha = 0$, then so too must be the ε_α. Equation (7a) fixes the dimensions of ε_α to be, $[\varepsilon_\alpha] = (\text{mass})^{-1/2}$ so that in (7b) we require the derivative $\not{\partial} \equiv \gamma^a\partial_a$ on dimensional grounds. We shall work with 4-component spinors $\psi_\alpha(x)$, ($\alpha = 1,2,3,4$) and for a minimal theory, restrict our attention to self conjugate Majorana spinors (see Appendix A) with four degrees of freedom. For Majorana spinors we raise and lower spinor indices with the antisymmetric charge conjugation matrix[4] $C_{\alpha\beta} = -\,C_{\beta\alpha}$ (Appendix B) and write the inner product of two such spinors as,

$$\underline{\bar{\varepsilon}\psi \equiv \varepsilon^\alpha\psi_\alpha \equiv \varepsilon_\alpha C^{\alpha\beta}\psi_\beta = \psi_\beta C^{\beta\alpha}\varepsilon_\alpha = \bar{\psi}\varepsilon \equiv \psi^+\gamma_0\varepsilon} \tag{8}$$

where we have used $\{\psi_\alpha, \varepsilon_\beta\} = 0$.

The closure properties of (7) are established by evaluating the commutators $(\delta_1\delta_2 - \delta_2\delta_1)$.

First for $A(x)$ we find,

$$(\delta_1\delta_2 - \delta_2\delta_1)A(x) = \delta_1(\bar{\varepsilon}_2\psi) - \delta_2(\bar{\varepsilon}_1\psi)$$

$$= (\bar{\varepsilon}_2\gamma^a\varepsilon_1 - \bar{\varepsilon}_1\gamma^a\varepsilon_2)\partial_a A(x).$$

But $\bar{\varepsilon}_1\gamma^a\varepsilon_2 = \varepsilon_1{}^T C\gamma^a\varepsilon_2 = -\varepsilon_2{}^T(C\gamma^a)^T\varepsilon_1$

and $(C\gamma^a)^T = C\gamma^a$ (Appendix B)

$$\therefore \quad \underline{\bar{\varepsilon}_1\gamma^a\varepsilon_2 = -\bar{\varepsilon}_2\gamma^a\varepsilon_1} \tag{9}$$

and so

$$\underline{(\delta_1\delta_2 - \delta_2\delta_1)A(x) = 2\,\bar{\varepsilon}_2\gamma^a\varepsilon_1\partial_a A(x)} \quad . \tag{10}$$

The right hand side of (10) is precisely the form of a translation of $A(x)$,

$$\underline{[i\,\xi^a\,P_a,\, A(x)] = \xi^a\,\partial_a\,A(x)} \tag{11}$$

where P_a is the 4-momentum generator of space-time translations.

For $\psi_\alpha(x)$ we find,

$$(\delta_1\delta_2 - \delta_2\delta_1)\psi = \delta_1\,\partial\!\!\!/ A\,\varepsilon_2 - (1 \leftrightarrow 2)$$

$$= \partial\!\!\!/(\bar{\varepsilon}_1\psi)\varepsilon_2 - (1 \leftrightarrow 2)\ .$$

Using the Fierz rearrangements (Appendix B),

$$\underline{\psi_\alpha\phi^\beta = \tfrac{1}{4}\{-\bar{\psi}\phi\delta_\alpha{}^\beta + \bar{\psi}\gamma_5\phi(\gamma_5)_\alpha{}^\beta + \bar{\psi}i\gamma^a\gamma_5\phi(i\gamma_a\gamma_5)_\alpha{}^\beta}$$

$$\underline{+\,\bar{\psi}\,\gamma^a\phi(\gamma_a)_\alpha{}^\beta + \tfrac{1}{2}\,\bar{\psi}\,\sigma^{ab}\phi(\sigma_{ab})_\alpha{}^\beta\}} \tag{12}$$

and the symmetry properties of the bilinears deduced from (9), we obtain,

$$(\delta_1\delta_2 - \delta_2\delta_1)\psi = \frac{1}{2}\bar{\varepsilon}_2\gamma^a\varepsilon_1\not{\partial}\gamma_a\psi + \frac{1}{4}\bar{\varepsilon}_2\sigma^{ab}\varepsilon_1\not{\partial}\sigma_{ab}\psi .$$

Hence,

$$\underline{(\delta_1\delta_2 - \delta_2\delta_1)\psi = \bar{\varepsilon}_2\gamma^a\varepsilon_1\partial_a\psi - \frac{1}{2}\bar{\varepsilon}_2\gamma^a\varepsilon_1\gamma_a\not{\partial}\psi}$$

$$\underline{- \frac{1}{4}\bar{\varepsilon}_2\sigma^{ab}\varepsilon_1\sigma_{ab}\not{\partial}\psi - \frac{i}{2}\bar{\varepsilon}_2\sigma^{ab}\varepsilon_1(\gamma_a\partial_b - \gamma_b\partial_a)\psi} . \qquad (13)$$

The first term on the right hand side is a translation term as in (10) the next two terms vanish if $\psi_\alpha(x)$ satisfies the massless free field equation, $\not{\partial}\psi = 0$, but the last term has no clear interpretation. We conclude that a scalar field $A(x)$ and a spin $\frac{1}{2}$ Majorana field $\psi_\alpha(x)$ don't provide a closed multiplet to represent the spinorial transformations of the form of equations (7).

The smallest multiplet is in fact obtained by adding a pseudo-scalar field $B(x)$ to the $A(x)$ and $\psi_\alpha(x)$, then modifying (7) to,

$$\underline{\delta A(x) = \bar{\varepsilon}\psi} \qquad (14a)$$

$$\underline{\delta B(x) = \bar{\varepsilon}\gamma_5\psi} \qquad (14b)$$

$$\underline{\delta\psi(x) = \not{\partial}(A + \gamma_5 B)\varepsilon} \qquad (14c)$$

The closure relations for (14) may be calculated as for (7), to obtain

$$\underline{(\delta_1\delta_2 - \delta_2\delta_1)A(x) = 2\bar{\varepsilon}_2\gamma^a\varepsilon_1\partial_a A(x)} \qquad (15a)$$

$$\underline{(\delta_1\delta_2 - \delta_2\delta_1) B(x) = 2\bar{\varepsilon}_2\gamma^a\varepsilon_1\partial_a B(x)} \qquad (15b)$$

$$(\delta_1\delta_2 - \delta_2\delta_1)\psi(x) = 2\bar{\varepsilon}_2\gamma^a\varepsilon_1\partial_a\,\psi(x) - \bar{\varepsilon}_2\gamma^a\varepsilon_1\gamma_a\not{\partial}\psi(x) \,. \tag{15c}$$

Hence the commutator of two of these spinorial transformations closes on the translations,

$$\delta\Phi = [i\,\xi^a P_a\,\Phi] = \xi^a\partial_a\Phi \qquad (\Phi \equiv \{A, B, \psi\})$$

with $\xi^a = 2\,\bar{\varepsilon}_2\,\gamma^a\varepsilon_1$, provided $\psi(x)$ satisfies the equation of motion, $\not{\partial}\psi = 0$. This unusual feature that the closure relations depend upon field equations means that we must specify an invariant action for the fields $A(x)$, $B(x)$ and $\psi(x)$ in order that they may form a closed multiplet. In fact it is not difficult to verify that the Lagrangian for the free fields,

$$\mathcal{L}_0 = -\tfrac{1}{2}(\partial_a A)^2 - \tfrac{1}{2}(\partial_a\,B)^2 - \frac{i}{2}\,\bar{\psi}\,\not{\partial}\psi \tag{16}$$

transforms only by a total divergence under (14). We find that,

$$\delta\,\mathcal{L}_0 = -\,\partial_a\left[\bar{\varepsilon}\psi\partial^a A + \bar{\varepsilon}\gamma^5\psi\partial^a B - \frac{1}{2}\,\bar{\varepsilon}\not{\partial}(A + \gamma_5 B)\gamma^a\psi\right] ,$$

hence $\int\mathcal{L}_0 d^4x$ is an invariant action. The dependence of the closure relations upon the equations of motion of the $\psi(x)$ field may be understood in terms of the degrees of freedom in the multiplet. Off-shell there are four Fermi degrees of freedom in the $\psi(x)$ and two Bose degrees of freedom, one each in the real $A(x)$ and $B(x)$. Now since the spinorial transformations are linear one-one (they are required to be one-one since they close on translations) transformations between Bose and Fermi fields we cannot expect closure unless we have an equal number of Bose and Fermi degrees of freedom. Hence the $\psi(x)$

must be constrained to two degrees of freedom by its field equations $\not\partial\psi = 0$. If we require off-shell closure, independent of any field equations it is clear that we must introduce two more Bose fields. Furthermore in order to cancel the unwanted term in (15c) these two fields, $F(x)$ and $G(x)$, should transform $\sim\not\partial\psi$. The multiplet $\{A, B, F, G, \psi\}$ obtained in this manner is known as the Wess-Zumino multiplet, named after its discoverers[76]. The transformations of the fields in this multiplet are,

$$\underline{\delta A(x) = \bar{\varepsilon}\psi} \tag{17a}$$

$$\underline{\delta B(x) = \bar{\varepsilon}\gamma_5\psi} \tag{17b}$$

$$\underline{\delta\psi(x) = \not\partial(A + \gamma_5 B)\varepsilon + F\varepsilon + i\,G\gamma_5\varepsilon} \tag{17c}$$

$$\underline{\delta F(x) = \bar{\varepsilon}\not\partial\psi} \tag{17d}$$

$$\underline{\delta G(x) = i\bar{\varepsilon}\not\partial\gamma_5\psi}\;. \tag{17e}$$

It is straightforward to check that the closure relations all take the form,

$$(\delta_1\delta_2 - \delta_2\delta_1)\Phi = [i\,\xi^a P_a, \Phi] = \xi^a\partial_a\Phi \tag{18}$$

where $\Phi \equiv \{A, B, F, G, \psi\}$ and $\xi^a = 2\,\bar{\varepsilon}_2\gamma^a\varepsilon_1$.

Notice that $[F] = [G] = (\text{mass})^2$ but this is not a problem since we do not wish the fields $F(x)$ and $G(x)$ to correspond to physical states. On-shell there are only two Fermion degrees of freedom so we still only require two Bose degrees of freedom. Hence the $F(x)$ and $G(x)$ must be auxiliary fields with algebraic equations of motion. The simplest action invariant under (17) is the free field action

(16) together with quadratic terms for the auxiliary fields,

$$\underline{\mathcal{L} = \mathcal{L}_0 + \tfrac{1}{2}(F^2 + G^2)} \qquad (19)$$

(note that the field equations for F and G are $F = G = 0$).
The variation of $\mathcal{L}$ under (17) is easily found to be,

$$\delta\mathcal{L} = \delta\mathcal{L}_0 - \partial_a(\tfrac{1}{2}\,\bar{\psi}\gamma^a(F + iG\gamma_5)\varepsilon)$$

so that $\int \mathcal{L}\, d^4x$ is an invariant action.

The spinorial transformations (17) were given the name 'supergauge transformations' by Wess and Zumino who also found larger multiplet representations(76). Auxiliary fields quite generally occur in off-shell supergauge multiplets because of the different number of constraints that field equations impose on Fermionic and Bosonic fields and the fact that any off-shell multiplet should reduce to an on-shell multiplet.

We define the generators S_α of the supergauge transformations by,

$$\underline{\delta_\varepsilon\Phi = [\bar{\varepsilon}S, \Phi]} \qquad (20)$$

Then the closure relation (18) reads,

$$\begin{aligned}(\delta_1\delta_2 - \delta_2\delta_1)\Phi &= [\bar{\varepsilon}_1 S, [\bar{\varepsilon}_2 S, \Phi]] - [\bar{\varepsilon}_2 S, [\bar{\varepsilon}_1 S, \Phi]] \\ &= [i\,\xi^a P_a, \Phi] \,.\end{aligned}$$

Using the Jacobi identities,

$$[A, [B,C]] + [B, [C,A]] + [C, [A,B]] \equiv 0$$

we obtain,

$$-[\Phi, [\bar{\varepsilon}_1 S, \bar{\varepsilon}_2 S]] = [i\,\xi^a P_a, \Phi]$$

hence we extract,

$$\underline{[\bar{\varepsilon}_1 S,\ \bar{\varepsilon}_2 S] \quad = \quad i\,\xi^a P_a \quad = \quad 2i\,\bar{\varepsilon}_2\,\gamma^a \varepsilon_1\,P_a} \tag{21}$$

But

$$\begin{aligned}[\bar{\varepsilon}_1 S,\ \bar{\varepsilon}_2 S] \quad &= \quad [\bar{\varepsilon}_1 S,\ \bar{S}\,\varepsilon_2] \\ &= \quad \bar{\varepsilon}_1{}^{\alpha}[S_\alpha, \bar{S}\varepsilon_2] + [\bar{\varepsilon}_1{}^{\alpha},\ \bar{S}\,\varepsilon_2]S_\alpha \\ &= \quad \bar{\varepsilon}_1{}^{\alpha}\{S_\alpha,\ \bar{S}^\beta\}\varepsilon_{2\beta} + 0 \quad .\end{aligned}$$

Hence,

$$\underline{\{S_\alpha, \bar{S}^\beta\} \quad = \quad 2i(\gamma^a)_\alpha{}^\beta\,P_a \quad \equiv \quad 2i(\not{P})_\alpha{}^\beta} \tag{22}$$

This relation shows how two spinorial generators S_α close, through anticommutators, on the translation generators P_a. The numerical factor $2i$ in (22) is not important, it may be traced to our choice in equations (14) of no overall numerical factor in the transformation of the field. It is the general structure, $\{S,\bar{S}\} \sim \not{P}$ which is the fundamental feature of the supergauge transformations. Since the supergauge transformations are global, $\varepsilon_\alpha \neq \varepsilon_\alpha(x)$ it is easy to see that they will commute with the translations, $\delta_\xi \Phi = \xi^a \partial_a \Phi$, i.e.

$$(\delta_\varepsilon \delta_\xi - \delta_\xi \delta_\varepsilon)\Phi \quad = \quad 0.$$

Then the Jacobi identities yield in this case,

$$\underline{[P_a,\ S_\alpha] \quad = \quad 0} \quad . \tag{23}$$

Equations (22) and (23) describe the supergauge algebra which is extended to the supersymmetry algebra when we introduce Lorentz group SO(1,3) generators $M_{ab} = -M_{ba}$ and supplement (22) and (23) with the Poincaré ISO(1,3) algebra (see equations (I.53)),

$$[M_{ab}, M_{cd}] = i(\eta_{ac}M_{bd} + \eta_{bd}M_{ac} - \eta_{ad}M_{bc} - \eta_{bc}M_{ad}) \quad (24a)$$

$$[M_{ab}, P_c] = i(\eta_{ac}P_b - \eta_{bc}P_a) \quad (24b)$$

$$[P_a, P_b] = 0 \quad (24c)$$

where η_{ab} is the Minkowski metric.

Since the supergauge transformations connect fields of different spin it is clear that they won't commute with the Lorentz transformations, rather that the commutator takes the form $[M,S] \subset S$. Furthermore the fields in the Wess-Zumino multiplet connected by S_α differ by spin $\frac{1}{2}$ so that S_α itself must belong to this particular representation of SO(1,3) and we write,

$$[M_{ab}, S_\alpha] = \frac{1}{2}(\sigma_{ab}\, S)_\alpha \quad (25)$$

Equations (22), (23), (24) and (25) constitute the supersymmetry or graded ISO(1,3) algebra. A graded Lie algebra(14) consists of 'odd' and 'even' generators (see section III.1 for more details) with the required structure,

$$[\text{even, even}] \subset \text{even}$$

$$[\text{even, odd}] \subset \text{odd}$$

$$\{\text{odd, odd}\} \subset \text{even}$$

For the supersymmetry algebra the even generators are the $\{P_a, M_{ab}\}$ which themselves form an ordinary Lie sub-algebra. The odd generators are the $\{S_\alpha\}$ which grade the ISO(1,3) Lie sub-algebra.

Given this algebraic structure of supersymmetry we may deduce some general features of its representations on multiples of particle states.

First we point out that since $[P_a, S_\alpha] = 0$ it follows that the quadratic Casimir invariant P^aP_a of ISO(1,3) is also a Casimir invariant of the supersymmetry group. Hence it follows that all the on-shell states in a supersymmetric multiplet must have the same mass, m_o, where $P^aP_a = m_o^2$. Supersymmetry therefore requires equal mass Fermions and Bosons.

The particle states in a supermultiplet, differing by spin $\frac{1}{2}$ are connected by the S_α,

$$S_\alpha \; |\text{Fermi}> \sim |\text{Bose}> , \qquad S_\alpha \; |\text{Bose}> \sim |\text{Fermi}> \quad .$$

We now proceed to identify the on-shell states in any irreducible supersymmetric multiplet[65].

This is achieved through the construction of the Fock space of states $|P_a, j>$ using the S_α as raising and lowering operators. Here j is a spin (helicity) label of massive (massless) states. The fact that the SO(3) subgroup of the Lorentz group is sufficient to label states is due to the on-shell constraint, $P^aP_a = m_o^2$. For $m_o \neq 0$ this spontaneously breaks SO(1,3) down to SO(3) so that we may fix a value of P_a (subject to $P^aP_a = m_o^2$) and label states according to the SO(3) subgroup. The full group representation is then obtained by applying Lorentz boosts (coset elements) to the fixed frame states[64].

For massive states it is convenient to work in the rest frame $P_a = (m_o, 0, 0, 0)$ so that equation (22) becomes,

$$\{S_\alpha, \overline{S}^\beta\} = 2i(\gamma^o)_\alpha{}^\beta \, m_o$$

or

$$\underline{\{S_\alpha, S_\beta\} = 2i(\gamma^o C)_{\alpha\beta} \, m_o} \qquad (26)$$

where, according to our conventions the Majorana condition on S_α reads, (Appendix B), $\bar{S}^\alpha = (S^T C)^\alpha$. Working in the representation of the γ matrices where γ^o is diagonal (Appendix B) and

$$C = i\gamma^2\gamma^o = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & 0 & -1 \\ & & 1 & 0 \\ \hline 0 & -1 & \multicolumn{2}{c}{0} \\ 1 & 0 & & \end{array}\right)$$

then the Majorana condition reads,

$$S = -C\bar{S}^T = -C\,\gamma^o S^* = -i\gamma^2 S^*$$

i.e.

$$\begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix} = -\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & 0 & 1 \\ & & -1 & 0 \\ \hline 0 & -1 & \multicolumn{2}{c}{0} \\ 1 & 0 & & \end{array}\right) \begin{pmatrix} S_1^* \\ S_2^* \\ S_3^* \\ S_4^* \end{pmatrix}$$

$$\Longrightarrow \qquad \underline{S_4 = -S_1^*} \qquad \text{and} \qquad \underline{S_3 = S_2^*}$$

In this representation we may describe the Majorana spinor S_α by the two component spinor S_μ $(\mu = 1,2)$ and its conjugate $S_\mu^* = (-S_4, S_3)$. Equation (26) therefore decomposes to

$$\{S_\mu, S_\nu\} = 0, \quad \{S_\mu^*, S_\nu^*\} = 0 \quad \text{and} \quad \{S_\mu, S_\nu^*\} = 2i\, m_o\, \delta_{\mu\nu}.$$

Now, for the SO(3) subgroup, (25) reduces to

$$[M_{ij}, S_\alpha] = \tfrac{1}{2}(\sigma_{ij} S)_\alpha \qquad (i,j = 1,2,3).$$

In Appendix A we show that $\sigma_{ij} = -\frac{1}{2}\varepsilon_{ijk} \left(\begin{array}{c|c} \sigma_\kappa & 0 \\ \hline 0 & \sigma_\kappa \end{array}\right)$

also $M_{ij} = -\frac{1}{2}\varepsilon_{ijk} J_k$, where J_k are the usual angular momentum SU(2) generators satisfying $[J_i, J_j] = i\varepsilon_{ijk} J_k$ and σ_i are the Pauli matrices.

Hence $$[J_i, S_\mu] = \frac{1}{2}(\sigma_i S)_\mu$$

so that S_μ and its conjugate S_μ^* are two component SU(2) spinor raising and lowering operators. A massive multiplet is therefore characterized by the common mass m_o and the maximum spin state $|J>$, the other states are then, $S_1^*|J>$, $S_2^*|J>$ and $S_1^* S_2^*|J>$ (the $S_1^*|J>$ and $S_2^*|J>$ have the same spin but different parity[64]). In the table below we list some of the massive multiplets, (read off horizontally across the table).

Massive Multiplets

Scalar	Pseudoscalar	Spin $\frac{1}{2}$	Vector	Pseudovector	Spin $\frac{3}{2}$
1	1	1			
1		2	1		
	1	2		1	
		1	1	1	1

For zero mass multiplets, $P^a P_a = 0$, and the stability subgroup is no longer SO(3). We cannot work in the rest frame and instead we choose $P_a = (P,0,0,P)$ which is invariant under the generators, M_{12}, $M_{01} + M_{13}$, $M_{02} + M_{23}$. Writing $M_{12} = -\frac{1}{2} J_3$, $M_{01} + M_{13} \equiv T_1$ and $M_{02} + M_{23} \equiv T_2$ we see from (24a) that these generators close on the algebra ISO(2),

$$[J_3, T_1] = iT_2$$

$$[J_3, T_2] = -iT_1$$

$$[T_1, T_2] = 0 .$$

As for the Poincaré group, the finite dimensional representations of this group are the representations of the orthogonal subgroup, in this case SO(2). Hence the massless states are characterized by the helicity, λ, where

$$J_3|\lambda\rangle = \lambda|\lambda\rangle \qquad (\lambda = 0, \pm\tfrac{1}{2}, \pm 1 \ldots).$$

For the massless case, equation (22) becomes,

$$\begin{aligned}\{S_\alpha, S_\beta\} &= 2i(\gamma^0 C + \gamma^3 C)_{\alpha\beta}\, P \\ &= -2i \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 1 & 0 & & \end{array}\right) P\end{aligned}$$

so that $\{S_\mu, S_\nu\} = \{S_\mu^*, S_\nu^*\} = 0$

$$\{S_1, S_2^*\} = \{S_2, S_1^*\} = \{S_2, S_2^*\} = 0$$

and $$\{S_1, S_1^*\} = 2i\, P.$$

Clearly S_2 'creates' zero norm unphysical states $(S_2 S_2^*|\rangle \equiv 0)$, also

$$[J_3, S_1] = \tfrac{1}{2}(\sigma_3 . S)_1 = \tfrac{1}{2} S_1 .$$

and $$[J_3, S_1^*] = -\tfrac{1}{2} S_1^* .$$

The Fock space of states therefore starts with a maximum helicity state $|\lambda\rangle$ and contains one other, $S_1^*|\lambda\rangle$ with helicity $\lambda - \frac{1}{2}$. Massless multiplets therefore contain only two different helicity states, $\pm\lambda$ and $\pm(\lambda - \frac{1}{2})$. We tabulate a few of these multiplets below.

Massless Multiplets

Scalar	Pseudoscalar	$\lambda = \frac{1}{2}$	$\lambda = 1$	$\lambda = \frac{3}{2}$	$\lambda = 2$
1	1	1			
		1	1		
				1	1

Extended Supersymmetry

So far we have only considered a Poincaré algebra graded by a single 4-component spinor generator S_α. This algebraic structure may be generalized to the Poincaré algebra, graded by N spinorial charges $S_\alpha^{\ A}$ $(A = 1,2,\ldots, N)$ to obtain an extended supersymmetry algebra. One immediately recognisable consequence of the application of this extended algebra to physics is that larger particle multiplets are obtained[24] since there are now N times as many Fock space raising and lowering operators. In the case of massless multiplets it is easy to see that with N spinor generators the helicity states range from λ to $\lambda - \frac{N}{2}$, hence if we wish to avoid states with helicities >2 then we are restricted to $N \leqslant 8$.

The algebraic structure of these extended supersymmetries is enriched by introducing an internal Lie symmetry group G of which the $(S_\alpha)^A$ form a basis for the N-dimensional representation. With T_i $(i = 1, \ldots, n \quad \dim G)$ as the generators of G we have,

$$[T_i, T_j] = f_{ij}^{\ \ k}\, T_k$$

and

$$\underline{[T_i, S_\alpha^{\ A}] = [T_i]^A_{\ B}\, S_\alpha^{\ B} \equiv f_{i\ B}^{\ A}\, S_\alpha^{\ B}} \tag{27}$$

where $[T_i]^A_{\ B} \equiv f_{i\ B}^{\ A}$ are the matrix elements of the N-dimensional representation of the generators T_i of G. The superalgebra therefore consists of the generators

$$X_I \equiv \{P_a, M_{ab}, S_\alpha{}^A, T_i\} \qquad (I = 1, \ldots, 10 + 4N + n)$$

with

$$[X_I, X_J\} = f_{IJ}{}^K X_K$$

($[A,B\}$ is an anticommutator for A,B both odd and a commutator otherwise). In particular the closure relation for the spinor generators takes the form,

$$\underline{\{S_\alpha{}^A, S_\beta{}^B\} = 2i\, \delta^{AB}(\gamma^a C)_{\alpha\beta}\, P_a .} \tag{28}$$

This relation was shown by Haag, Lopuszanski and Sohnius[37] to be the most general closure relation for the $S_\alpha{}^A$ consistent with the observed symmetries of the S-matrix. The only possible generalization is to add, to the right hand side, the term,

$$U^{AB}\, C_{\alpha\beta} + V^{AB}(\gamma^5 C)_{\alpha\beta}$$

where the $U^{AB} = -U^{BA}$ and $V^{AB} = -V^{BA}$ are central charges in the superalgebra, (i.e. $[U^{AB}, X_I] = [V^{AB}, X_I] = [U^{AB}, V^{AB}] = 0$), formed from the T_i generators. Further analysis by Haag et al.[37] showed that in the case of a massless theory ($P^a P_a = 0$) then the algebra could be extended to that of conformal supersymmetry[24] with an internal symmetry which is required (by Jacobi identities) to be SU(N) or a subgroup. For the extended Poincaré supersymmetry there is no such restriction on the internal symmetry group G, however SO(N) appears as a 'natural'[24] choice, particularly in the context of extended supergravity theories with gauged internal symmetries, (see section III.2).

Extended supersymmetries have generated much interest because they provide a unique framework in which to combine internal and space-time symmetries in a non-trivial way, through equations (27). They are therefore the only symmetry groups known which could conceivably describe a unified theory of the interactions in nature. The problems, (particularly the one of symmetry breaking), however are many and we, in fact, shall devote our attention in later chapters to the minimal $N = 1$ case with no internal symmetry group.

Supersymmetric Gauge Theories

In this section we shall briefly consider supersymmetric theories with gauged internal symmetries. These are not the same internal symmetry groups as we discussed in the extended supersymmetries section which rotated the spinor generators into one another. Here we consider the direct product of the supersymmetry group with the gauge group. The simplest example is provided by, supersymmetry $\otimes$ U(1), to give a supersymmetric theory of massless (and chargeless!) Q.E.D. The action for this theory is,

$$\underline{\mathcal{L}_o = -\tfrac{1}{4}(F_{ab})^2 - \tfrac{i}{2}\,\bar{\psi}\,\not{\partial}\,\psi} \tag{29}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ and $A_a(x)$ is the vector potential for the electromagnetic field. It is clear that $\mathcal{L}_o$ is invariant under the U(1) gauge transformations,

$$\underline{\delta_\omega A_a = \partial_a\,\omega(x)} \tag{30a}$$

$$\underline{\delta_\omega \psi = 0} \quad . \tag{30b}$$

In addition we may easily verify that $\mathcal{L}_o$ transforms only by a total divergence under the supersymmetry transformations,

$$\underline{\delta_\varepsilon A^a = i\,\bar{\varepsilon}\,\gamma^a\,\psi} \tag{31a}$$

and

$$\underline{\delta_\varepsilon \psi = \tfrac{i}{2}\,F^{ab}\,\sigma_{ab}\,\varepsilon} \quad . \tag{31b}$$

In fact, we find that

$$\delta_\varepsilon \mathcal{L}_o = - \partial_a(i \bar{\varepsilon} \gamma_b \psi F^{ab} + \frac{1}{4} \bar{\varepsilon} \sigma^{bc} \gamma^a \psi F_{bc}) \; .. \tag{32}$$

Let us examine the supersymmetry transformations (31) in more detail. The most general form, on dimensional grounds, that these transformations may take is,

$$\delta_\varepsilon A^a = i \bar{\varepsilon} \gamma^a \psi \tag{33a}$$

and

$$\delta_\varepsilon \psi = a \partial^a A_a \varepsilon + bF^{ab} \sigma_{ab} \varepsilon \tag{33b}$$

where 'a' and 'b' are just numbers.

The closure relations are then,

$$(\delta_1\delta_2 - \delta_2\delta_1)A^a = i \bar{\varepsilon}_2 \gamma^a(a\partial^b A_b \varepsilon_1 + b \sigma^{cd} F_{cd}\varepsilon_1) - (1 \leftrightarrow 2) \; .$$

Hence,

$$(\delta_1\delta_2 - \delta_2\delta_1)A_a = 2ia \partial^b A_b \bar{\varepsilon}_2\gamma_a\varepsilon_1 - 4b \partial_a(\bar{\varepsilon}_2 \not{A} \varepsilon_1) + 4b \bar{\varepsilon}_2 \gamma^b \varepsilon_1 \partial_b A_a \tag{34}$$

The last term corresponds to a translation but we have no clear interpretation as yet for the first two terms. Before discussing them further we calculate the closure relations for the $\psi(x)$ and find,

$$(\delta_1\delta_2 - \delta_2\delta_1)\psi = 4b \bar{\varepsilon}_2 \gamma^a \varepsilon_1 \partial_a\psi + (\tfrac{i}{2}a-b) \bar{\varepsilon}_2 \gamma^a \varepsilon_1 \gamma_a \not{\partial} \psi + (\tfrac{i}{4}a + \tfrac{b}{2}) \bar{\varepsilon}_2 \sigma^{ab} \varepsilon_1 \sigma_{ab} \not{\partial} \psi \; . \tag{35}$$

Hence, supersymmetry transformations of $\psi(x)$ close on translations provided $\psi(x)$ satisfies its field equation $\not{\partial}\psi = 0$. Returning now to (34) we may understand the closure properties of this relation only when we consider the gauge transformations (30) in addition to the supersymmetry transformations. First, notice that the requirement that the supersymmetry and internal U(1) transformations commute implies that,

$$(\delta_\omega \delta_\varepsilon - \delta_\varepsilon \delta_\omega)\Phi = 0 \qquad \text{for} \quad \Phi = \{A^a, \psi\} .$$

For A^a this relation holds identically but for ψ we find

$$(\delta_\omega \delta_\varepsilon - \delta_\varepsilon \delta_\omega)\psi = a(\partial^b \partial_b \omega)\varepsilon$$

and since $\partial^b \partial_b \omega(x) \neq 0$ in general, we set $\underline{a = 0}$.

We therefore drop the first term in (34) and recognize that the second term is simply a gauge transformation,

$$\delta_\omega A^a = \partial_a \omega(A) \qquad \text{with} \qquad \omega(A) = -4b\, \bar{\varepsilon}_2 \not{A}\, \varepsilon_1 .$$

The fact that we require the gauge transformations (30) in addition to the field equations to close the algebra for this multiplet should not be surprising since it is the gauge invariance which reduces the degrees of freedom of the A^a to two on-shell, (i.e. the same as $\psi(x)$, as required).

Off-shell there are four Fermi degrees of freedom and three Bose degrees of freedom, (A^a still loses one degree of freedom as we factor out gauge equivalent classes). It follows therefore that we should attain off-shell closure with the addition of a single auxiliary Bose field D(x) transforming $\sim \not{\partial}\psi$. In fact D(x) is required to be a pseudoscalar field and the off-shell transformations become,

$$\underline{\delta_\varepsilon A^a = i\, \bar{\varepsilon}\, \gamma^a\, \psi} \tag{36a}$$

$$\underline{\delta_\varepsilon \psi = \frac{i}{2} F^{ab} \sigma_{ab} \varepsilon + D\gamma_5 \varepsilon} \tag{36b}$$

$$\underline{\delta_\varepsilon D = i\, \bar{\varepsilon}\, \gamma_5\, \not{\partial}\, \psi} \quad . \tag{36c}$$

An invariant action for this multiplet is $\int \mathcal{L}\, d^4x$, where

$$\mathcal{L} = \mathcal{L}_o + \frac{1}{2} D^2$$

and

$$\delta_\varepsilon \mathcal{L} = \delta_\varepsilon \mathcal{L}_o + \partial_a (\tfrac{i}{2} D(x)\, \bar{\varepsilon}\, \gamma_5 \gamma^a \psi) \quad .$$

The generalization of this U(1) theory to non-abelian Yang-Mills theories is not difficult and a supersymmetric model may be set up along analogous lines to that described above. Supersymmetry requires the Fermi fields to belong to the adjoint multiplet of the gauge group, (for equal numbers of Fermi and Bose degrees of freedom), so that the ordinary derivative in (29) must be replaced by a gauge covariant derivative. The remaining analysis proceeds as before and we obtain on-shell closure onto translations and gauge transformations. For off-shell closure, an adjoint multiplet of auxiliary Bose fields is required.

Finally, we point out that if we require the gauge group and supersymmetry to be combined non trivially so that

$$(\delta_\omega \delta_\varepsilon - \delta_\varepsilon \delta_\omega) \neq 0$$

or, through the Jacobi identities,

$$[T_i, S_\alpha] \neq 0.$$

Then since an even and an odd generator close on an odd generator we require,

$$[T, S] \subset S$$

and we are therefore led to consider extended supersymmetries,

$$[T_i, S_\alpha{}^A] = [T_i]^A{}_B\, S_\alpha{}^B \quad .$$

However, with local gauge transformations $\omega(x)^i T_i$, it follows from the above commutator that even if the spinor parameters $\varepsilon_\alpha^{\ A}$ are constants before a gauge transformation they will certainly be space-time dependent afterwards. In conclusion we state that it is possible to have a non-trivial combination of global supersymmetry and global internal symmetries <u>or</u> a trivial, direct product, combination of global supersymmetry with local internal symmetries <u>but</u> a non-trivial combination of supersymmetry with local internal symmetries forces local supersymmetry. Local supersymmetry is the subject of supergravity theory which we shall discuss in section 3. Before that, however, we mention a few other features of global supersymmetry.

<u>Spontaneous Supersymmetry Breaking</u>

One of the most striking features of supersymmetry is the mass degeneracy between Fermions and Bosons in a given multiplet. This degeneracy is also an undesirable feature, however, since it is not observed in nature. Clearly supersymmetry must be broken, either explicitly (thereby losing all its advantages) or spontaneously. Quite generally, spontaneous symmetry breaking is triggered by the non-vanishing of the vacuum expectation values (determined by a potential) of some of the fields in a multiplet representation of a group G which is consequently broken to the stability subgroup H. Goldstone's theorem then guarantees $n = \dim \frac{G}{H}$ massless Goldstone modes and the remaining fields with the non-vanishing vacuum expectation values may be massive in general. The spontaneous breaking of the supersymmetry group to the Poincaré group, with the S_α spinors as the broken generators should be accompanied by the emergence of a massless Goldstone Fermion ψ with $\langle\delta\psi\rangle_0 \neq 0$. Now for supersymmetry multiplets the general form for the vacuum expectation value of $\delta\psi$ is

$$<\delta\psi>_0 \quad \sim \quad <\text{auxiliary fields}>_0$$

since the physical Bose fields occur in $\delta\psi$ as derivatives. For example, for the Wess-Zumino multiplet from (17) we see that,

$$<\delta\psi>_0 \quad = \quad <(F + iG\gamma_5)>_0 \, \varepsilon$$

and for the Q.E.D. multiplet (36),

$$<\delta\psi>_0 \quad = \quad <D>_0 \, \gamma_5 \, \varepsilon \ .$$

Hence supersymmetry is spontaneously broken only if one or more of the auxiliary fields in a multiplet acquires a non-vanishing vacuum expectation value. This however is difficult to achieve. Recall that the auxiliary fields are related to the physical fields by algebraic field equations which may be solved for $\Phi_i(A_\mu)$ where Φ_i are the auxiliary field and A_μ the physical fields. On substituting this back into any supersymmetric potential the form of the resulting potential is quite generally[(24),(60)] $\sim (\Phi_i(A_\mu))^2$ and hence minimises at $\Phi_i = 0$ where supersymmetry is unbroken. The only escape from this conclusion occurs when the system of equations $\Phi_i(A_\mu) = 0$ has no solution and although this is possible[(24)] it is highly model dependent. Supersymmetry is therefore much more difficult to break spontaneously than internal symmetries.

The weak point in our presentation of supersymmetry so far is that we have no systematic tensor calculus for the supersymmetry multiplets which enables us to combine them and form invariants as candidates for supersymmetric actions. We therefore close this section with an outline of the most general approach to this problem, the superfield method.

Superfields on Superspace

Minkowski space-time is isomorphic to the coset space $\frac{ISO(1,3)}{SO(1,3)}$ which is the set of all translations $e^{i\xi^a P_a}$, with parameters ξ^a. The vector space of translation parameters $\{\xi^a\}$ is then isomorphic to that of the Lorentz coordinates $\{x^a\}$.

Superspace is an extension of Minkowski space-time, isomorphic to the coset space, $\frac{\text{Graded } ISO(1,3)}{SO(1,3)}$, which is the set of all translations $e^{i\xi^a P_a}$ and supergauge transformations $e^{\bar{\varepsilon}S}$ where $e^{\bar{\varepsilon}S} e^{i\xi.P} = e^{i\xi.P} e^{\bar{\varepsilon}S}$. Superspace may therefore be parametrized by $\{x^a, \theta_\alpha\}$ where $\{\theta_\alpha\}$ form a 4-dimensional Grassmann algebra isomorphic to that of the ε_α. An element, g of the supersymmetry group may be written,

$$\underline{g = e^{\bar{\theta}S} e^{i\xi.P} h \equiv U(x,\theta)h} \qquad (37)$$

where $h \in SO(1,3)$, the stability subgroup of any point in superspace. The (linear) realization of supersymmetry on superspace is now easily obtained using the supersymmetry algebra. We find that,

$$e^{\bar{\varepsilon}S}\, U(x,\theta) \equiv e^{\bar{\varepsilon}S} e^{\bar{\theta}S} e^{ix.P} = U(x + i\bar{\varepsilon}\gamma\theta,\ \theta+\varepsilon)$$

and

$$e^{i\xi.P}\, U(x,\theta) \equiv e^{i\xi.P} e^{\bar{\theta}S} e^{ix.P} = U(x+\xi,\theta)$$

and

$$e^{i\omega.M}\, U(x,\theta) = U(\Lambda(\omega)x,\ a(\omega)\theta)e^{i\omega.M}$$

where $i\omega M \equiv \frac{i}{2}\omega^{ab}M_{ab}$, $\Lambda(\omega)^a{}_b \equiv e^{\omega^a{}_b}$ is the 4×4 Lorentz matrix representation and $a(\omega)_\alpha{}^\beta \equiv (e^{\frac{i\omega}{4}^{ab}(\sigma_{ab})})_\alpha{}^\beta$ the Dirac spinor representation of $SO(1,3)$.

The action of the supersymmetry group on superspace is thus:

Supergauge Translations, ε_α: $\quad x^a \to x'^a = x^a + i\,\bar{\varepsilon}\,\gamma^a\theta$

$$\theta_\alpha \to \theta_\alpha' = \theta_\alpha + \varepsilon_\alpha$$

Spatial Translations, ξ^a: $\quad x^a \to x'^a = x^a + \xi^a$

$$\theta_\alpha \to \theta_\alpha' = \theta_\alpha$$

Lorentz Translations, ω^{ab}: $\quad x^a \to x'^a = \Lambda(\omega)^a{}_b \, x^b$

$$\theta_\alpha \to \theta_\alpha' = a(\omega)_\alpha{}^\beta \, \theta_\beta.$$

Now a function f on a real manifold M is a mapping

$$f : M \to R.$$

A general coordinate system on an n-dimensional differentiable manifold is a set of n functions x^μ $(\mu = 1,\ldots, n)$ so that each point $P \in M$ is labelled by n real numbers. Lorentz coordinates $x^a(P)$ $(a = 1,2,3,4)$ labelling points in Minkowski space-time are used to evaluate fields (which are also functions) such as the scalar field $\phi(P) = \phi(x^a(P)) \equiv \phi(x^a)$. Under Lorentz coordinate transformations $\Phi(P)$ must of course be unaffected so that $\phi(x^a) = \phi'(x'^a)$.

The above analysis, though rather laboured from a physicist's point of view is useful in clarifying the definition of a superfield on superspace. The coordinates $(x^a(P), \theta_\alpha(P))$ of a point P in superspace are defined through a set of 8 coordinate functions x^a, θ_α which map superspace <u>not</u> to $\mathbb{R}^8$ but to $\mathbb{R}^4 \otimes G^4$ where G^4 is the 4-dimensional Grassmann algebra. Superspace is thus a manifold, locally similar to $\mathbb{R}^4 \otimes G^4$ which, although it has conceptual problems, may have functions, vector fields, differential forms, affine connections, etc., defined on it, (see for example ref. (55)).

In particular we may define a real function Φ on superspace to be a mapping,

$$\Phi \; : \; \text{superspace} \to \mathbb{R}$$

i.e. $\Phi(P) = \Phi(x^a(P), \theta_\alpha(P)) \equiv \Phi(x,\theta) \in \mathbb{R}$.

Obviously $\Phi(P)$ is independent of the coordinates employed in its evaluation so that,

$$\Phi(x^a(P),\ \theta_\alpha(P)) = \Phi'(x'^a(P),\ \theta_\alpha'(P)) \equiv \Phi(P) .$$

Then, for a supergauge transformation, we require

$$\Phi(x^a,\theta_\alpha) = \Phi'(x^a + i\,\bar{\varepsilon}\,\gamma^a\theta,\ \theta_\alpha+\varepsilon_\alpha)$$

so that

$$\begin{aligned}\Phi'(x^a,\theta_\alpha) &= \Phi(x^a - i\,\bar{\varepsilon}\,\gamma^a\theta,\ \theta_\alpha - \varepsilon_\alpha)\\ &= \Phi(x^a,\theta_\alpha) - i\,\bar{\varepsilon}\gamma^a\theta\partial_a\Phi\,, - \varepsilon_\alpha\frac{\partial}{\partial\theta_\alpha}\Phi + O(\varepsilon^2) .\end{aligned}$$

Hence with,

$$\delta_\varepsilon\Phi \equiv \Phi'(x,\theta) - \Phi(x,\theta) = [\bar{\varepsilon}S,\Phi]$$

we obtain the superfield realization of the generators S,

$$\underline{[S_\alpha,\ \Phi] = -\,(\frac{\partial}{\partial\bar{\theta}}{}_\alpha + i(\gamma^a\theta)_\alpha\ \partial_a)\Phi} \quad . \tag{38}$$

Similarly we find, for the translations,

$$\underline{[P_a,\ \Phi] = -\,i\,\partial_a\,\Phi} \tag{39}$$

and for the Lorentz rotations,

$$\underline{[M_{ab},\ \Phi] = (i(x_a\partial_b - x_b\partial_a) + \tfrac{1}{2}\,\bar{\theta}\,\sigma^{ab}\,\frac{\partial}{\partial\bar{\theta}})\,\Phi} \ . \tag{40}$$

The value of superfields for constructing field multiplet representations of supersymmetry rests upon the fact that the Fermi coordinates θ_α are Grassmann variables so that $\theta^5 \equiv 0$. Hence any superfield (x,θ) has an exact fourth order Taylor expansion in the θ_α coordinates, the coefficients of which are fields on Minkowski space-time. Quite generally

$$\Phi(x,\theta) = \phi(x) + \phi^{\alpha}(x)\theta_{\alpha} + \frac{1}{2}\phi^{[\alpha\beta]}(x)\,\theta_{\alpha}\theta_{\beta} + $$

$$+ \frac{1}{6}\phi^{[\alpha\beta\gamma]}(x)\theta_{\alpha}\theta_{\beta}\theta_{\gamma} + \frac{1}{24}\phi^{[\alpha\beta\gamma\delta]}(x)\theta_{\alpha}\theta_{\beta}\theta_{\gamma}\theta_{\delta}$$

where

$$\phi(x) \equiv \Phi(x,0)$$

$$\phi^{\alpha}(x) \equiv \frac{\partial}{\partial\theta_{\alpha}}\Phi(x,\theta)\Big|_{\theta_{\alpha}=0}$$

etc.

Using the six antisymmetric matrices C, $C\gamma_5$ and $C\gamma_a\gamma_5$, the Taylor expansion may be rewritten,

$$\underline{\Phi(x,\theta) = A(x)+\bar{\theta}\psi(x) + \frac{1}{4}\bar{\theta}\theta\, B(x) + \frac{1}{4}\bar{\theta}\gamma_5\theta\, D(x) +}$$

$$\underline{+ \frac{1}{4}\bar{\theta}\,\gamma^{a}\gamma_5\theta\, A_a(x) + \frac{1}{4}\bar{\theta}\theta\,\bar{\theta}\,\chi(x) + \frac{1}{32}(\bar{\theta}\theta)^2\, F(x)} \qquad (41)$$

Hence a scalar superfield contains,

3 scalar fields $A(x)$, $B(x)$ and $F(x)$,

1 pseudoscalar field $D(x)$

2 Majorana fields $\psi(x)$ and $\chi(x)$

1 pseudovector field $A_a(x)$.

By construction these fields form a closed supersymmetry multiplet with the transformations given by (38) ,

$$\delta_{\varepsilon}\Phi = -\bar{\varepsilon}\left(\frac{\partial}{\partial\bar{\theta}} + i\gamma^{a}\theta\,\partial_a\right)\Phi \quad .$$

In component form this equation reads,

$$\delta_{\varepsilon} A(x) = -\bar{\varepsilon}\,\psi$$

$$\delta_{\varepsilon}\psi(x) = -\frac{1}{2}(B + D\gamma_5 + i\gamma^{a}\gamma_5 A_a - 2i\,\not{\partial} A)\varepsilon$$

$$\delta_{\varepsilon} B(x) = -\frac{1}{2}\bar{\varepsilon}\,\chi + i\,\bar{\varepsilon}\,\not{\partial}\,\psi$$

$$\delta_{\varepsilon} D(x) = -\frac{1}{2}\bar{\varepsilon}\,\gamma_5\chi + i\,\bar{\varepsilon}\,\gamma_5\not{\partial}\,\psi$$

$$\delta_{\varepsilon} A_a(x) = -\frac{1}{2}\bar{\varepsilon}\,\gamma_a\gamma_5\chi + \bar{\varepsilon}\,\gamma^{b}\gamma_a\gamma_5\partial_b\psi$$

$$\delta_\varepsilon \chi(x) = \frac{1}{2}(-F + 2i \not\partial B + 2i\not\partial\gamma_5 D + 2\gamma_5\gamma^a \not\partial A_a)\varepsilon$$

$$\delta_\varepsilon F(x) = 2i \bar{\varepsilon} \not\partial \chi .$$

This is the pseudovector multiplet[(24), (65)], named after its highest spin field.

Finally we list some general features of the superfield approach.

(a) By construction, superfield supermultiplets always give off-shell representations of supersymmetry, hence they always contain auxiliary fields.

(b) In addition to the scalar superfield $\Phi(x,\theta)$ described above we may define superfields $\Psi_i(x,\theta)$ belonging to any representation of the Lorentz group and transforming as,

$$\Psi_i'(x',\theta') = M(\omega)_i{}^j \Psi_j(x,\theta)$$

where $M(\omega)_i{}^j$ is an $n \times n$ matrix representation of $SO(1,3)$.

(c) A superfield covariant derivative[(65)] D_α may be defined by,

$$D_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} - i(\gamma^a\theta)_\alpha \partial_a$$

so that,

$$[\bar{\varepsilon}S, D_\alpha] = 0$$

and

$$[M_{ab}, D_\alpha] = \frac{1}{2}(\sigma_{ab} D)_\alpha .$$

Hence from a scalar superfield Φ we can construct a spinor superfield $D_\alpha\Phi$ with the same supersymmetry transformation properties as Φ. These covariant derivatives are useful for constructing constrained superfields and invariants (see below).

(d) Two superfield supermultiplets may be multiplied together to give a new supermultiplet,

$$\Phi_1\Phi_2 \equiv \Phi_{12} = A_{12}(x) + \bar{\theta}\psi_{12}(x) + \frac{1}{4}\bar{\theta}\theta B_{12}(x) + \ldots..$$

where $A_{12}(x) \;=\; A_1(x)\,A_2(x)$

$$\psi_{12}(x) \;=\; A_1(x)\psi_2(x) + A_2(x)\psi_1(x)$$

$$B_{12}(x) \;=\; A_1(x)B_2(x) + A_2(x)B_1(x) - \bar{\psi}_1(x)\psi_2(x) \;.$$

etc.

(e) Invariant actions may be formed by acting on a superfield (itself a product of other superfields) with the covariant derivative four times. This ensures that all the θ dependent terms in the resultant superfield are space-time divergences, so that $\delta_\varepsilon((\bar{D}D)^2\Phi) \;=\;$ total divergence.

(f) The big drawback with the superfield multiplets is that they are generally reducible and must be constrained by relations such as, $(1+\gamma_5)D_\alpha\Phi \;=\; 0$ which projects out an eight component chiral superfield[65] from the sixteen component Φ. Modulo the problem of constraints, superfields have proven of much value in constructing supersymmetric actions, finding the auxiliary fields for known on-shell multiplets (in particular for supergravity[68]) and in evaluating the quantum effects in supersymmetric theories through the supergraph techniques[35].

3) Supergravity

A supersymmetric theory of gravitation must involve a multiplet containing the massless spin 2 graviton. If we are to avoid spins > 2 then we see from the analysis in section 2 that for a minimal supersymmetric theory the relevant on-shell multiplet should consist of a graviton with helicity ± 2 and a gravitino with helicity $\pm \frac{3}{2}$. The free field action for the spin 2 field $\phi_{ab} = \phi_{ba}$ was given in equation (5) and yields the classical field equations (6) for the ϕ_{ab},

$$\partial^c\partial_c\,\phi_{ab} - \partial^c\partial_a\,\phi_{bc} - \partial^c\partial_b\,\phi_{ac} - \partial_a\partial_b\,\phi^c{}_c \;=\; 0\;.$$

It is easy to verify that these equations are invariant under the gauge transformations,

$$\underline{\delta_\xi\,\phi_{ab} \;=\; \partial_a\,\xi_b + \partial_b\,\xi_a}\;. \tag{42}$$

It follows that we may always choose a gauge so that $\partial^b\phi_{ab} = \frac{1}{2}\partial_a\phi^b{}_b$ and in which the field equations consequently reduce to the massless Klein-Gordan equation,

$$\partial^c\partial_c\,\phi_{ab} \;=\; 0\;,$$

as we should expect for a Bose field.

The spin $\frac{3}{2}$ field is contained in the symmetric rank 3 spinor field $\Psi_{\alpha\beta\gamma}(x)$, where $\alpha, \beta, \gamma = 1, 2, 3, 4$ are Dirac spinor indices (see Appendix A). This Ψ may be rewritten using the symmetric Dirac matrices,

$$\Psi_{\alpha\beta\gamma} \;=\; (C\gamma^a)_{\alpha\beta}\,(\psi_a)_\gamma + \tfrac{1}{2}(C\sigma^{ab})_{\alpha\beta}\,(\lambda_{ab})_\gamma\;.$$

The requirement that the right hand side is symmetric under $\alpha \leftrightarrow \gamma$ and $\beta \leftrightarrow \gamma$ interchange and that Ψ satisfies the massless Dirac equation on each of its indices (e.g., $\not\partial_\alpha{}^\delta\,\Psi_{\delta\beta\gamma} = 0$) then determines $(\lambda_{ab})_\alpha$ in terms of $(\psi_a)_\alpha$ and constrains $(\psi_a)_\alpha$ to satisfy the Rarita-Schwinger equations (see for example ref. (53)),

$$\underline{\not\partial\,\psi_a \;=\; 0} \tag{43a}$$

and

$$\underline{\gamma^a\,\psi_a \;=\; 0} \tag{43b}$$

Notice that these two equations then imply

$$\partial^a \psi_a = 0 . \qquad (43c)$$

Equation (43a) is simply the massless Dirac equation for the vector-spinor field $\psi_{a\alpha}$. The remaining two equations serve to remove the spin $\frac{1}{2}$ components of the Clebsch-Gordan decomposition, $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$, implicit in the definition of $\psi_{a\alpha}$.

These Rarita-Schwinger equations for $\psi_{a\alpha}$ may be combined into the single equation

$$R^a \equiv \varepsilon^{abcd} \gamma_5 \gamma_b \partial_c \psi_d = 0 . \qquad (44)$$

Notice that R^a is invariant under the gauge transformation

$$\delta_\varepsilon \psi_{a\alpha} = \partial_a \varepsilon_\alpha \qquad (45)$$

where $\varepsilon_\alpha = \varepsilon_\alpha(x)$ is a local spinorial parameter.

We therefore see that equation (43b), $\gamma^a \psi_a = 0$ emerges here as a gauge fixing condition. Also

$$\begin{aligned} \gamma^a R_a &= i\, \varepsilon^{abcd} \gamma_5 \sigma_{ab} \partial_c \psi_d \\ &= -2i\, \sigma^{cd}\, \partial_c \psi_d \\ &= \not{\partial}\, \gamma^a \psi_a - \gamma^a \not{\partial}\, \psi_a \end{aligned}$$

hence in the gauge $\gamma^a \psi_a = 0$ then the equation $R_a = 0$ implies $\not{\partial} \psi_a = 0$, which is equation (43a).

In constructing a supermultiplet of the two fields ϕ_{ab} and $\psi_{a\alpha}$ it is clear from our previous discussion of supersymmetric Yang-Mills theories that we must take into account the local gauge symmetries,

$$\delta_\xi \phi_{ab} = \partial_a \xi_b + \partial_b \xi_a$$

and

$$\delta_\varepsilon \psi_{a\alpha} = \partial_a \varepsilon_\alpha .$$

The parameters $\xi_b(x)$ have the form of local translation or infinitesimal general coordinate transformation $x^a \to x'^a = x^a + \xi^a$ parameters. Indeed, when the free spin 2 action is derived from the Einstein action (as was done in section 1) then the gauge invariance $\delta\phi_{ab} = \partial_a\xi_b + \partial_b\xi_a$ is simply the first order (i.e. neglecting $O(\xi^2)$) part of the general coordinate invariance of the theory[57]. The parameters $\varepsilon_\alpha(x)$ have the form of local supergauge parameters and with this interpretation we see that the action for free spin 2 and spin $\frac{3}{2}$ fields is invariant under the 'supergauge transformations',

$$\delta_\varepsilon \phi_{ab} = 0 \quad \text{and} \quad \delta_\varepsilon \psi_{a\alpha} = \partial_a \varepsilon_\alpha .$$

With the identification of $\phi_{ab}(x)$ as the graviton field then its quadratic free field action is necessarily only a part of the full Einstein action, $\mathcal{L}_E = \frac{1}{\kappa^2}\sqrt{-g}\, R$. The Rarita-Schwinger field may be coupled to gravity via the formation of the appropriate Lorentz covariant derivatives as outlined in section I.4. In curved space-time the Rarita-Schwinger action should, according to our minimal coupling prescription, take the form

$$\mathcal{L}_{R-S} = h\, \varepsilon^{abcd}\, \bar{\psi}_a \gamma_5 \gamma_b \nabla_c \psi_d = -\varepsilon^{\mu\nu\kappa\lambda}\, \bar{\psi}_\mu \gamma_5 \gamma_\nu \nabla_\kappa \psi_\lambda$$

where

$$\nabla_\mu \psi_{d\alpha} = \partial_\mu \psi_{d\alpha} + \frac{i}{4} B_\mu{}^{ab} \sigma_{ab} \psi_{d\alpha} - B_{\mu d}{}^{a}\, \psi_{a\alpha}$$

or

$$\nabla_\mu \psi_{\nu\alpha} = h_\nu{}^{a} \nabla_\mu \psi_{a\alpha} = \partial_\mu \psi_{\nu\alpha} + \frac{i}{4} B_\mu{}^{ab} \sigma_{ab} \psi_{\nu\alpha} - \Gamma_{\mu\nu}{}^{\kappa}\, \psi_{\kappa\alpha} .$$

A consistent set of supersymmetry transformations, mixing the vierbein and Rarita-Schwinger fields and under which the sum of the Einstein and Rarita-Schwinger actions is invariant, may only be found if the Rarita-Schwinger action is modified to a non-minimal coupling[31].

Freedman, van Nieuwenhuizen and Ferrara[30] and independently, Deser and Zumino[16] found supersymmetry transformations under which the sum of the Einstein action and modified Rarita-Schwinger action was invariant. This so called supergravity Lagrangian takes the form,

$$\mathcal{L}_{SG} = \frac{1}{\kappa^2} h\, h^\mu{}_a h^\nu{}_b R_{\mu\nu}{}^{ab} - 2i\, \varepsilon^{\mu\nu\kappa\lambda}\, \bar{\psi}_\mu \gamma_5 \gamma_\nu D_\kappa \psi_\lambda \tag{46}$$

where $D_\kappa \psi_\lambda = \partial_\kappa \psi_\lambda + \frac{i}{4} B_\kappa{}^{ab} \sigma_{ab} \psi_\lambda$ is just the spinor covariant derivative, with the vector index 'λ' ignored. In this sense the Rarita-Schwinger field is non-minimally coupled to gravity.

The supersymmetry transformations under which $\mathcal{L}_{SG}$ is invariant are

$$\delta_\varepsilon\, h_\mu{}^a = i\, \kappa\, \bar{\varepsilon} \gamma^a \psi_\mu \tag{47a}$$

and

$$\delta_\varepsilon\, \psi_{\mu\alpha} = \frac{1}{\kappa} (\partial_\mu \varepsilon_\alpha + \frac{i}{4} B_\mu{}^{ab} (\sigma_{ab} \varepsilon)_\alpha) \equiv \frac{1}{\kappa} D_\mu \varepsilon_\alpha \tag{47b}$$

where κ is the gravitational coupling constant.

Notice that although $I_{SG} \equiv \int d^4x\, \mathcal{L}_{SG} = I_{SG}\left[h_{\mu a}, \psi_\mu, B_\mu{}^{ab}\right]$ we have not any transformation for $B_\mu{}^{ab}$ in the above set. Recall that the $B_\mu{}^{ab}$ satisfy the algebraic field equations (I.43) which may be solved to give $B_\mu{}^{ab}(h,\psi)$. For the particular case of the supergravity action we find (in Appendix C) that

$$B_\mu{}^{ab}(h,\psi) = B_\mu{}^{ab}(h) - \frac{i}{2} \kappa^2 (\bar{\psi}_\mu \gamma^a \psi^b + \bar{\psi}^a \gamma^b \psi_\mu + \bar{\psi}^a \gamma_\mu \psi^b)$$

where $B_\mu{}^{ab'}(h)$ is the torsion-free connection (I.44).

Now in evaluating the variation of I_{SG} we may either treat the $B_\mu{}^{ab}$ as independent fields (first order formalism) or substitute their equations of motion $B_\mu{}^{ab} = B_\mu{}^{ab}(h,\psi)$ leaving only the $h_\mu{}^a$ and ψ_μ as the independent fields (second order formalism). The variation of I_{SG} is written in the first order formalism as

$$\delta \mathcal{L}_{SG} = \left.\frac{\delta I_{SG}}{\delta h_{\mu a}}\right|_{\psi,B} \delta h_{\mu a} + \left.\frac{\delta I_{SG}}{\delta \psi_\mu}\right|_{h,B} \delta\psi_\mu + \left.\frac{\delta I_{SG}}{\delta B_\mu{}^{ab}}\right|_{h,\psi} \delta B_\mu{}^{ab} ,$$

however, the last term in this expression vanishes identically when $B_\mu{}^{ab}$ satisfies its equation of motion so that, quite generally,

$$\delta \mathcal{L}_{SG} = \left.\frac{\delta I_{SG}}{\delta h_{\mu a}}\right|_{\psi,B(h,\psi)} \delta h_{\mu a} + \left.\frac{\delta I_{SG}}{\delta \psi_\mu}\right|_{h,B(h,\phi)} \delta\psi_\mu \tag{48}$$

This variational principle is not simply the second order formalism since the $B_\mu{}^{ab}$ is held fixed in both terms. It has consequently been named the '1.5 order formalism'(58). The proof of the invariance of I_{SG} under the local supersymmetry transformations (47) is relatively straightforward and we shall now give a brief outline.

With $I_{SG} = I_E + I_{RS}$ then (48) becomes

$$\delta_\varepsilon \mathcal{L}_{SG} = \frac{\delta I_E}{\delta h_{\mu a}} \delta_\varepsilon h_{\mu a} + \frac{\delta I_{RS}}{\delta h_{\mu a}} \delta_\varepsilon h_{\mu a} + \frac{\delta I_{RS}}{\delta \psi_\mu} \delta_\varepsilon \psi_\mu \tag{49}$$

with $\delta_\varepsilon h_{\mu a} = i\kappa \bar{\varepsilon}\gamma_a\psi_\mu$

and $\delta\psi_\mu = \frac{1}{\kappa} D_\mu \varepsilon.$

Using the variation of the Einstein action, calculated in Appendix C then we find that

$$\frac{\delta I_E}{\delta h_{\mu a}} \delta_\varepsilon h_{\mu a} = \frac{2h}{\kappa} G_{\mu a}\, i\, \bar{\varepsilon}\gamma^a\psi^\mu = -\frac{2ih}{\kappa} G_{\mu a}\, \bar{\varepsilon}\gamma^\mu\psi^a \tag{50a}$$

(where $G_{\mu a} = R_{\mu a} - \frac{1}{2} h_{\mu a} R)$.

Also

$$\frac{\delta I_{RS}}{\delta h_{\mu a}} \delta h_{\mu a} = 2\kappa\, \varepsilon^{\mu\nu\kappa\lambda}\, \bar{\varepsilon}\gamma^a\psi_\nu \bar{\psi}_\mu \gamma_5\gamma_a D_\kappa\psi_\lambda$$

then using the Fierz resummation relation (12) on the $\psi_\nu\bar\psi_\mu$ product and the identities,

$$\gamma^a\gamma_b\gamma_5\gamma_a = -2\gamma_5\gamma_b$$

and

$$\gamma^a\sigma_{bc}\gamma_5\gamma_a = 0$$

we find

$$\frac{\delta I_{RS}}{\delta h_{\mu a}}\ \delta_\varepsilon h_{\mu a} = -\kappa\, \varepsilon^{\mu\nu\kappa\lambda}\, \bar\varepsilon\, \gamma_5\gamma_b D_\kappa\, \psi_\lambda\bar\psi_\nu\gamma^b\psi_\mu \quad . \tag{50b}$$

Finally,

$$\frac{\delta I_{RS}}{\delta\psi_\mu}\ \delta_\varepsilon\psi_\mu = -\frac{2i}{\kappa}\ \varepsilon^{\mu\nu\kappa\lambda}\left[\overline{D_\mu\varepsilon}\ \ \gamma_5\gamma_\nu\,{}_\kappa\psi_\lambda + \bar\psi_\mu\gamma\ \gamma_\nu D_\kappa D_\lambda\ \varepsilon\right]$$

$$= \text{total derivative} + \frac{2i}{\kappa}\ \varepsilon^{\mu\nu\kappa\lambda}\left[\bar\varepsilon\ \gamma_5(D_\mu e_\nu{}^a)\gamma_a D_\kappa\psi_\lambda + \right.$$

$$\left.\bar\varepsilon\gamma_5\gamma_\nu D_\mu D_\kappa\psi_\lambda - \bar\psi_\mu\gamma_5\gamma_\nu D_\kappa D_\lambda\ \varepsilon\right]$$

Now $\quad [D_\mu, D_\nu]\varepsilon = \frac{i}{4} R_{\mu\nu}{}^{ab}\sigma_{ab}\ \varepsilon$

and $\quad [D_\mu, D_\nu]\psi_\lambda = \frac{i}{4} R_{\mu\nu}{}^{ab}\sigma_{ab}\ \psi_\lambda$

so that, using the identity,

$$\{\gamma_\nu,\sigma_{ab}\} = h_\nu{}^c(-2i\ e_{cabd}\ \gamma^d\gamma_5)$$

the last two terms in the variation are combined to give,

$$\frac{i}{2\kappa}\ \varepsilon^{\mu\nu\kappa\lambda}\ \varepsilon_{abcd}\ h_\nu{}^c\ \bar\varepsilon\ \gamma^d\psi_\mu\ B_{\kappa\lambda}{}^{ab}$$

but (Appendix C)

$$\varepsilon^{\mu\nu\kappa\lambda}\ \varepsilon_{abcd}\ h_\nu{}^c = -h\ \ h^\mu{}_{[a}h^\kappa{}_b\ \ h^\lambda{}_{d]}$$

so that the term above becomes,

$$\frac{2ih}{\kappa} \bar{\varepsilon} \gamma^{\mu} \psi^{a} G_{\mu a}$$

and cancels (50a).

The remaining term in the ψ_μ variation is then combined with (50b) to give

$$\delta_\varepsilon \mathcal{L}_{SG} = \frac{2i}{\kappa} \varepsilon^{\mu\nu\kappa\lambda} \bar{\varepsilon} \gamma_5 \gamma_a D_\kappa \psi_\lambda (D_\mu h_\nu{}^a - \frac{i}{2} \kappa^2 \bar{\psi}_\mu \gamma^a \psi_\nu).$$

In Appendix C we show that the algebraic field equations for the $B_\mu{}^{ab}$ are,

$$D_\mu h_\nu{}^a - D_\nu h_\mu{}^a = i \kappa^2 \bar{\psi}_\mu \gamma^a \psi_\nu \quad . \qquad (51)$$

Hence,

$$\underline{\delta_\varepsilon \mathcal{L}_{SG} = 0,} \quad \text{(up to a total derivative).}$$

Having established some interest in the transformations,

$$\delta_\varepsilon h_{\mu a} = i \kappa \bar{\varepsilon} \gamma_a \psi_\mu$$

$$\delta_\varepsilon \psi_\mu = \frac{1}{\kappa} D_\mu \varepsilon$$

as describing a local Fermi-Bose symmetry in the supergravity action $\mathcal{L}_{SG}$, it is natural to ask whether these transformations realize a closed algebraic structure in the same manner as the supergauge transformations on flat space-time. For the vierbein fields we find,

$$(\delta_1\delta_2 - \delta_2\delta_1) h_{\mu a} = i \bar{\varepsilon}_2 \gamma_a D_\mu \varepsilon_1 - i \bar{\varepsilon}_1 \gamma_a D_\mu \varepsilon_2$$

$$= D_\mu (i \bar{\varepsilon}_2 \gamma_a \varepsilon_1).$$

The bilinear $i \bar{\varepsilon}_2 \gamma_a \varepsilon_1$ is of the same form as that which occurred in the global supersymmetry closure relations, though now it corresponds to a local translation parameter, $\xi_a(x) = i\,\bar{\varepsilon}_2\gamma_a\varepsilon_1$. To understand further this closure relation, recall that an infinitesimal general coordinate transformation, $x^\mu \to x^\mu + \xi^\mu$, produces the following transformation on a world vector V_μ,

$$\delta V_\mu \equiv V_\mu'(x') - V_\mu(x) = -(\partial_\mu \xi^\nu) V_\nu .$$

onsidered actively this transformation reads,

$$\delta_o V_\mu \equiv V_\mu'(x) - V_\mu(x) = -(\partial_\mu \xi^\nu) V_\nu - \xi^\nu \partial_\nu V_\mu .$$

Returning to the closure relation,

$$(\delta_1\delta_2 - \delta_2\delta_1) h_{\mu a} = D_\mu \xi_a = \partial_\mu \xi_a - B_{\mu a}{}^b \xi_b$$

$$= \partial_\mu (h_{\nu a}\xi^\nu) - \xi^\nu (B_{\mu ab} h_\nu{}^b - B_{\nu ab} h_\mu{}^b) - \xi^\nu B_{\nu ab} h_\mu{}^b$$

$$= \xi^\nu \partial_\nu h_{\mu a} + (\partial_\mu \xi^\nu) h_{\nu a} + \xi^\nu (D_\mu h_{\nu a} - D_\nu h_{\mu a}) - \xi^\nu B_{\nu ab} h_\mu{}^b .$$

Then using the torsion condition (51),

$$\underline{(\delta_1\delta_2 - \delta_2\delta_1) h_{\mu a} = \xi^\nu \partial_\nu h_{\mu a} + (\partial_\mu \xi^\nu) h_{\nu a} + \xi^\nu\, i\,\kappa^2\, \bar{\psi}_\mu \gamma_a \psi_\nu}$$
$$\underline{- \xi^\nu B_{\nu a}{}^b h_{\mu b}} . \qquad (52)$$

Local supergauge transformations therefore close on

(a) General coordinate transformations, considered actively.

(b) Local supergauge transformations with parameters,

$$\bar{\varepsilon}_{12} = -\kappa\, \xi^\nu \bar{\psi}_\nu .$$

(c) Local Lorentz transformation with parameters,

$$\omega_{ab} = \xi^\nu B_{\nu ab} .$$

For the Rarita-Schwinger fields we find that

$$(\delta_1\delta_2 - \delta_2\delta_1)\psi_\mu = \frac{i}{4\kappa}\,\sigma_{ab}(\varepsilon_2\delta_{\varepsilon_1} B_\mu{}^{ab} - \varepsilon_1\delta_{\varepsilon_2} B_\mu{}^{ab})$$

Here we must calculate $\delta_\varepsilon B_\mu{}^{ab}$ using $B_\mu{}^{ab} = B_\mu{}^{ab}(h,\psi)$

then after a Fierz resummation to put ε_1 and ε_2 in the same bilinear covariant we find[(31)],

$$(\delta_1\delta_2 - \delta_2\delta_1)\psi_\mu = \xi^\nu(D_\nu\psi_\mu - D_\mu\psi_\nu)$$

+ terms proportional to the ψ_μ equations of motion

(see Appendix C for the supergravity field equations).

For on-shell closure we need only consider the term given explicitly and rewrite it to obtain,

$$\underline{(\delta_1\delta_2 - \delta_2\delta_1)\psi_\mu = \xi^\nu\partial_\nu\psi_\mu + (\partial_\mu\,\xi^\nu)\psi_\nu}$$

$$\underline{-\frac{1}{\kappa}(\partial_\mu(\kappa\,\xi^\nu\psi_\nu) + \frac{i}{4}B_\mu{}^{ab}\sigma_{ab}(\kappa\,\xi^\nu\psi_\nu))}$$

$$\underline{+\frac{i}{4}\,\xi^\nu\,B_\nu{}^{ab}\,\sigma_{ab}\,\psi_\mu} \tag{53}$$

The on-shell closure of two local supergauge transformations on ψ_μ is thus the same as that on $h_\mu{}^a$ with identical local translation, supergauge and Lorentz rotation parameters. Not surprisingly we have found that local supersymmetry mixes with the local symmetries of the gravitational field. The form of the transformations (47) also suggests that they could be part of a set of gauge transformations (with the inhomogeneous $\partial_\mu\varepsilon$ term) of an extended gauge theory of gravity. In the first chapter we examined the gauge symmetries of gravity and concluded that although translations could be viewed as gauge symmetries with the vierbein fields as corresponding gauge fields, this symmetry could only be realized non-linearly in a spontaneously broken gauge theory. It is the supersymmetric extension of this broken gauge theory of gravity which is the subject of the next chapter.

One final remark which should be made here is that, just as we found for flat space multiplets, the supergravity multiplet only closes when the spinor field (ψ_μ) satisfies its equation of motion On-shell there are two Bose (helicity ± 2) and two Fermi (helicity $\pm \frac{3}{2}$) physical degrees of freedom. Off-shell the number of degrees of freedom of the $\psi_{\mu a}$ would be $4 \times 4 = 16$. However, the local super-gauge invariance factors out 4 degrees of freedom. For the $h_{\mu a}$ there are $4 \times 4 = 16$ but then local Lorentz and general coordinate invariance factor out 10 degrees of freedom. Hence for off-shell closure of the local supergauge transformations at least six auxiliary Bose fields are required. This minimal set of auxiliary fields has been found[68] and consists of a scalar, a pseudoscalar and a pseudo-vector field, (see for example ref. (58) for more details).

CHAPTER III

SUPERGRAVITY AS A GAUGE THEORY OF OSp(1,4)

Field theories on flat Minkowski space-time may possess, in addition to their required Poincaré invariance, a global supersymmetric invariance. This global symmetry is realized as a set of transformations, parametrized by space-time independent spinors, ε_α, on a multiplet of Bose and Fermi fields. Explicit examples of such multiplets were discussed in the previous chapter.

A supersymmetric theory involving the graviton, however, may only be found if the spinorial transformation is space-time dependent, $\varepsilon_\alpha = \varepsilon_\alpha(x)$. It is then natural to ask whether this supergravity theory is the gauge theory of supersymmetry. We shall examine this gauge theory[9] and find that it has the same defect as the Poincaré gauge theory, (Chapter I), namely the identification of the translational gauge potentials as the Vierbein fields of the Einstein-Cartan theory of gravity. This then leads us to consider the group OSp(1,4) in a direct extension of the work done in the first chapter.

It will be convenient at the outset to list some general definitions and relations for gauge theories of supergroups.

1) General Formalism

A Lie supergroup, G, is a group, the elements of which are obtained by exponentiating its associated Graded Lie Algebra[14], $\mathcal{G}$.

We are only interested in Z_2 GLA's, with two types of element, generally called 'odd' and 'even' elements. $\mathcal{G}$ is a graded vector space which decomposes into the direct sum,

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1 \tag{1}$$

where, $\mathcal{G}_o$: aggregate of all even elements

$\mathcal{G}_1$: aggregate of all odd elements.

Let $\{X_i\}$ be a basis for $\mathcal{G}$, $(i = 1,2,\ldots,N;\ N = \dim \mathcal{G})$, then the closure of the algebra is described by,

$$\underline{(X_i, X_j) \equiv X_i X_j - (-1)^{\sigma_i \sigma_j} X_j X_i = f_{ij}{}^k X_k} \tag{2}$$

where $\sigma_i = \begin{cases} 0 & \text{for } X_i \text{ an even generator} \\ 1 & \text{for } X_i \text{ an odd generator} \end{cases}$.

$f_{ij}{}^k$ are the structure constants which clearly have the symmetry

$$\underline{f_{ij}{}^k = -(-1)^{\sigma_i \sigma_j} f_{ji}{}^k} \tag{3}$$

By definition(14) this Z_2 GLA must satisfy the decomposition,

$$\underline{(\mathcal{G}_o, \mathcal{G}_o) \subset \mathcal{G}_o} \tag{4a}$$

$$\underline{(\mathcal{G}_o, \mathcal{G}_1) \subset \mathcal{G}_1} \tag{4b}$$

$$\underline{(\mathcal{G}_1, \mathcal{G}_1) \subset \mathcal{G}_o} \quad . \tag{4c}$$

Notice that (4a) requires that the even elements must form a subalgebra and hence exponentiate to form an ordinary Lie subgroup. In terms of the basis $\{X_i\} = \{X_A, X_\alpha\}$, where X_A are the even generators and X_α the odd generators, then equations (2) and (4) imply,

$$\underline{[X_A, X_B] = f_{AB}{}^C X_C} \tag{5a}$$

$$\underline{[X_A, X_\alpha] = f_{A\alpha}{}^\beta X_\beta} \tag{5b}$$

$$\{X_\alpha, X_\beta\} = f_{\alpha\beta}{}^{A} X_A \tag{5c}$$

$$f_{A\beta}{}^{\alpha} = f_{A\alpha}{}^{\beta} = f_{\alpha\beta}{}^{\gamma} = 0 \quad . \tag{5d}$$

Given this GLA structure, of which the supersymmetry algebra of Chapter II is one example, we may proceed in the standard way (see Appendix to Ch. 1) to define the potentials and field strengths of a gauge theory of G:

Connection Form, ω on the principal fibre bundle, $P(V_4,G)$ is,

$$\omega \equiv \omega_\mu dx^\mu = \omega_\mu{}^{i} X_i \, dx^\mu = (\omega_\mu{}^{A} X_A + \omega_\mu{}^{\alpha} X_\alpha) dx^\mu \tag{6}$$

$\omega_\mu{}^{i}$ are the gauge potentials, the $\omega_\mu{}^{\alpha}$ components are Grassmann variables, $\{\omega_\mu{}^{\alpha}, \omega_\mu{}^{\beta}\} = 0$.

Covariant Derivative, ∇_μ is the 'horizontal lift' of ∂_μ and given, in direct product basis $\{\partial_\mu, X_i\}$ of $P(V_4,G)$ by,

$$\nabla_\mu = \partial_\mu - \omega_\mu{}^{i} X_i \quad . \tag{7}$$

∇_μ is, by definition (see appendix to Ch. 1) required to satisfy,

$$[\nabla_\mu, X_i] = 0 \quad .$$

Let $\varepsilon \in \mathcal{G}$ and write $\varepsilon = \varepsilon^A X_A + \varepsilon^\alpha X_\alpha$, with $\{\varepsilon^\alpha, \varepsilon^\beta\} = 0$. The transformation of ω_μ under infinitesimal local gauge transformations is given by,

$$\delta_\varepsilon \omega_\mu = -\delta_\varepsilon \nabla_\mu = -[\varepsilon^i X_i, \nabla_\mu]$$

$$= -[\varepsilon^i, \nabla_\mu] X_i \quad (\text{since } [X_i, \nabla_\mu] = 0)$$

$$= +(\nabla_\mu \varepsilon^i) X_i = [(\partial_\mu - \omega_\mu{}^j X_j)\varepsilon^i] X_i$$

$$= (\partial_\mu \varepsilon^i + f_{jk}{}^i \varepsilon^j \omega_\mu{}^k) X_i$$

i.e. $$\underline{\delta_\varepsilon \omega_\mu{}^i = \partial_\mu \varepsilon^i + f_{jk}{}^i \varepsilon^j \omega_\mu{}^k} \qquad (8)$$

Curvature Components, $F_{\mu\nu}{}^i$ are defined by,

$$\underline{[\nabla_\mu, \nabla_\nu] = -F_{\mu\nu}{}^i X_i} \qquad (9a)$$

so that

$$\underline{F_{\mu\nu}{}^i = \partial_\mu \omega_\nu{}^i - \partial_\nu \omega_\mu{}^i - f_{jk}{}^i \omega_\mu{}^j \omega_\nu{}^k} \qquad (9b)$$

and,

$$\underline{\delta_\varepsilon F_{\mu\nu}{}^i = f_{jk}{}^i \varepsilon^j F_{\mu\nu}{}^k} \qquad (10)$$

Finally we mention that the Jacobi identities for ∇_μ

$$[\nabla_\mu, [\nabla_\nu, \nabla_\kappa]] + [\nabla_\nu, [\nabla_\kappa, \nabla_\mu]] + [\nabla_\kappa, [\nabla_\mu, \nabla_\nu]] \equiv 0$$

imply, using the definition (9a),

$$\underline{\nabla_\mu F_{\nu\kappa}{}^i + \nabla_\nu F_{\kappa\mu}{}^i + \nabla_\kappa F_{\mu\nu}{}^i \equiv 0} \qquad (11)$$

Equations (11) are known as the Bianchi Identities for the $F_{\mu\nu}{}^i$.

2) <u>Gauge Theory of Supersymmetry</u>

The supersymmetry group is a supergroup whose GLA is the algebra of the Poincaré group, iso(1, 3), graded with the spinorial generators, S_α. Collecting together the relations from the previous two chapters we have,

$$\{X_i\} = \{X_A, X_\alpha\} = \{M_{ab}, P_a, S_\alpha\} \quad \text{and}$$

$$[M_{ab}, M_{cd}] = i(\eta_{ac} M_{bd} + \eta_{bd} M_{ac} - \eta_{ad} M_{bc} - \eta_{bc} M_{ad}) \tag{12a}$$

$$[M_{ab}, P_c] = i(\eta_{ac} P_b - \eta_{bc} P_a) \tag{12b}$$

$$[P_a, P_b] = 0 \tag{12c}$$

$$[M_{ab}, S_\alpha] = \tfrac{1}{2}(\sigma_{ab} S)_\alpha \tag{12d}$$

$$[P_a, S_\alpha] = 0 \tag{12e}$$

$$\{S_\alpha, S_\beta\} = (\gamma^a c)_{\alpha\beta} P_a \quad . \tag{12f}$$

Hence, we identify the structure constants for this supersymmetry algebra to be,

$$f_{ab\,cd}{}^{ef} = i(\eta_{ac}\delta^{[ef]}_{bd} + \eta_{bd}\delta^{[ef]}_{ac} - \eta_{ad}\delta^{[ef]}_{bc} - \eta_{bc}\delta^{[ef]}_{ad}) \tag{13a}$$

$$f_{abc}{}^{d} = i(\eta_{ac}\delta_b{}^d - \eta_{bc}\delta_a{}^d) \tag{13b}$$

$$f_{ab\alpha}{}^{\beta} = \tfrac{1}{2}(\sigma_{ab})_\alpha{}^\beta \tag{13c}$$

$$f_{\alpha\beta}{}^{a} = (\gamma^a c)_{\alpha\beta} \tag{13d}$$

and <u>all other $f_{ij}{}^k = 0$.</u>

The connection form for this group is written,

$$\omega_\mu = \frac{i}{2} B_\mu^{\ ab} M_{ab} + i h_\mu^{\ a} P_a + \bar{\psi}_\mu^{\ \alpha} S_\alpha \tag{14}$$

defining the gauge potentials, $B_\mu^{\ ab}$ of the SL(2,c) subgroup

$h_\mu^{\ a}$ translational potentials

$\psi_{\mu\alpha}$ supergauge potentials.

Let $A \in$ alg of supersymmetry group and write,

$$\underline{A = \frac{i}{2} \omega^{ab} M_{ab} + i\zeta^a P_a + \bar{\varepsilon}^{\alpha} S_\alpha} \ . \tag{15}$$

The transformation of the gauge potentials is given by (8) so that,

$$\delta_A \, i \, B_\mu^{\ ab} = \partial_\mu \, i\omega^{ab} + f_{ij}^{\ \ ab} \, \varepsilon^i \, \omega_\mu^{\ j}$$

then, using (13) we obtain,

$$\underline{\delta_A B_\mu^{\ ab} = \partial_\mu \omega^{ab} - B_{\mu\ c}^{\ a} \omega^{cb} - B_{\mu\ c}^{\ b} \omega^{ac} \equiv D_\mu \omega^{ab}}. \tag{16a}$$

Similarly we find

$$\underline{\delta_A h_\mu^{\ a} = D_\mu \zeta^a + \omega^a_{\ b} h_\mu^{\ b} + i \, \bar{\varepsilon} \gamma^a \psi_\mu} \tag{16b}$$

and

$$\underline{\delta_A \psi_{\mu\alpha} = (D_\mu \varepsilon)_\alpha - \frac{i}{4} \omega^{ab} (\sigma_{ab} \psi_\mu)_\alpha} \ . \tag{16c}$$

Notice, in particular that the supergauge transformations, with

$\omega^{ab} = \zeta^a = 0$, are,

$$\underline{\delta_\varepsilon B_\mu^{\ ab} = 0} \tag{17a}$$

$$\underline{\delta_\varepsilon h_\mu^{\ a} = i \, \bar{\varepsilon} \gamma^a \psi_\mu} \tag{17b}$$

$$\underline{\delta_\varepsilon \psi_\mu = D_\mu \varepsilon} \quad . \tag{17c}$$

These are the transformation of the supergravity fields, eqns. (2.47) in the '1-5 order formalism' where the spin connection compts, $B_\mu{}^{ab}$ are taken to satisfy their algebraic equations of motion, considered as non-dynamical constraints. (In order to recover eqns. (2.47) it is necessary to rescale $\psi_\mu \to \kappa\psi_\mu$ so that ψ_μ acquires the canonical dimensions, $(\mathrm{mass})^{3/2}$). Setting $\delta_\varepsilon B_\mu{}^{ab} = 0$ was justified in the 1.5 order formalism only because a non zero variation will not affect the variation of the supergravity action; here it follows as a direct consequence of the algebraic structure, (12). We may write down the field strengths using eqns. (9b) and (13),

$$F_{\mu\nu}{}^k X_k \equiv \frac{i}{2} B_{\mu\nu}{}^{ab} M_{ab} + i\, h_{\mu\nu}{}^a P_a + \bar{\psi}_{\mu\nu}{}^\alpha S_\alpha$$

and

$$\underline{B_{\mu\nu}{}^{ab} = \partial_\mu B_\nu{}^{ab} - \partial_\nu B_\mu{}^{ab} + (B_\mu{}^{ac} B_\nu{}^b{}_c - B_\nu{}^{ac} B_\mu{}^b{}_c)} \equiv R_{\mu\nu}{}^{ab} \tag{18a}$$

$$\underline{h_{\mu\nu}{}^a = D_\mu h_\nu{}^a - D_\nu h_\mu{}^a - i\, \bar{\psi}_\mu \gamma^a \psi_\nu} \tag{18b}$$

$$\underline{\psi_{\mu\nu\alpha} = (D_\mu \psi_\nu)_\alpha - (D_\nu \psi_\mu)_\alpha} \tag{18c}$$

We showed, in Chapter II, how the supergravity Lagrangian

$$\mathcal{L}_{SG} = \frac{1}{\kappa^2} h\, h^\mu{}_a h^\nu{}_b R_{\mu\nu}{}^{ab} - i\, \varepsilon^{\mu\nu\kappa\lambda} \bar{\psi}_\mu \gamma_5 \gamma_\nu \psi_{\kappa\lambda}$$

is invariant (up to a total divergence) under the transformations, eqns. (17). The translational field strengths, $h_{\mu\nu}{}^a$ don't occur in $\mathcal{L}_{SG}$ but, in fact provide the torsion constraint,

$$h_{\mu\nu}{}^a = D_\mu h_\nu{}^a - D_\nu h_\mu{}^a - i\, \psi_\mu \gamma^a \psi_\nu = 0.$$

Chamseddine and West[9] were among the first to consider supergravity as a gauge theory of supersymmetry. They pointed out that the invariance of the $B_\mu{}^{ab}$ under local supergauge transformations, (17a), is consistent with their algebraic field equations only when the torsion constraint, $h_{\mu\nu}{}^a = 0$ holds. They also showed that the most general Lagrangian linear in the field strengths, $R_{\mu\nu}{}^{ab}$ and $\psi_{\mu\nu\alpha}$, which reduces to the Einstein Lagrangian when $\psi_{\mu\alpha} \to 0$ and which is invariant under (17) is the supergravity Lagrangian, $\mathcal{L}_{SG}$.

It is important to notice, however, that $\mathcal{L}_{SG}$ is <u>not</u> invariant under the full group:

For pure 'translations', with $\omega^{ab} = \bar{\varepsilon}^\alpha = 0$,

$$\delta_\xi B_\mu{}^{ab} = 0 \tag{19a}$$

$$\delta_\xi h_\mu{}^a = D_\mu \zeta^a \tag{19b}$$

$$\delta_\xi \psi_{\mu\alpha} = 0 \quad . \tag{19c}$$

Hence,

$$\delta_\xi \mathcal{L}_{SG} = \frac{1}{4\kappa^2} \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{abcd}\, 2(D_\mu \xi^a) h_\nu{}^b R_{\kappa\lambda}{}^{cd}$$
$$- \varepsilon^{\mu\nu\kappa\lambda} \bar{\psi}_\mu \gamma_5 (D_\nu \xi^a) \gamma_a \psi_{\kappa\lambda} \quad .$$

Now carry out partial integration on both terms and use the Jacobi identities, (11), for the $R_{\kappa\lambda}{}^{cd}$:- $\varepsilon^{\mu\nu\kappa\lambda} D_\mu R_{\kappa\lambda}{}^{cd} \equiv 0$, to obtain

$$\delta_\xi \mathcal{L}_{SG} = \text{total derivative} - \frac{1}{2\kappa^2} \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{abcd}\, \xi^a (D_\mu h_\nu{}^b) R_{\kappa\lambda}{}^{cd}$$
$$+ i\, \varepsilon^{\mu\nu\kappa\lambda}\, \overline{D_\mu \psi_\mu}\, \gamma_5 \xi^a \gamma_a \psi_{\kappa\lambda} + i\, \varepsilon^{\mu\nu\kappa\lambda} \bar{\psi}_\mu \gamma_5 \xi^a \gamma_a 2 D_\nu D_\kappa \psi_\lambda \quad .$$

Substitution of the torsion constraint into first term and using,

$$[D_\nu, D_\kappa]\psi_\lambda = \frac{i}{4} R_{\nu\kappa}{}^{ab} \sigma_{ab} \psi_\lambda$$

in last term show, after a little algebra, that first and last terms cancel so that we are left with,

$$\underline{\delta_\xi \mathcal{L}_{SG} = \text{total derivative} - \frac{i}{2} \varepsilon^{\mu\nu\kappa\lambda} \bar{\psi}_{\mu\nu} \gamma_5 \xi^a \gamma_a \psi_{\kappa\lambda}} . \qquad (20)$$

Hence the supergravity action is not invariant under the full group, only under local Lorentz and local supergauge transformations. Short of abandoning the Einstein part of the action, we have no clear means of modifying the action in order to obtain translational invariance without losing the supergauge invariance of supergravity theory. We know, however, from the discussion in Ch. I, that the identification of the Vierbein fields as the gauge fields of local translations requires a more careful analysis and the recognition that gravity is a spontaneously broken gauge theory, with the translations as the broken generators.

It was, in fact, a gauge theory of the anti de Sitter group, SO(2,3), spontaneously broken to the Lorentz group, SO(1,3), which appeared to provide the correct geometrical description. For the remainder of this chapter we shall be concerned with the group OSp(1,4) which is the minimal graded extension of SO(2,3). Hereafter we shall refer to the supersymmetry group of flat space-time, eqns. (12), as the 'Wess-Zumino group'.

3) OSp(1,4) - Graded Anti de Sitter Group(80)

The anti de Sitter group, SO(2,3), is by definition, the group of linear transformations which leaves invariant, the quadratic form,

$$x^A x_A \equiv \eta_{AB} x^A x^B \quad . \qquad (A,B = 0,1,2,3,5)$$

where $\eta_{AB} = \mathrm{diag}(+1, -1, -1, -1, +1)$.

With $S \in SO(2,3)$ we shall write

$$S = \exp(\tfrac{i}{2}\ \omega^{AB}\ M_{AB}) .$$

The ten generators, $M_{AB} = - M_{BA}$, satisfy the usual orthogonal group algebra,

$$\underline{[M_{AB}, M_{CD}] = i(\eta_{AC} M_{BD} + \eta_{BD} M_{AC} - \eta_{AD} M_{BC} - \eta_{BC} M_{AD})} \tag{21}$$

or,

$$\underline{[M_{AB}, M_{CD}] = \tfrac{1}{2} F_{ABCD}{}^{EF} M_{EF}}$$

with

$$\underline{f_{AB}{}^{EF}{}_{CD} = i(\eta_{AC}\ \delta^{[EF]}_{BD} + \eta_{BD}\ \delta^{[EF]}_{AC} - \eta_{AD}\ \delta^{[EF]}_{BC} - \eta_{BC}\ \delta^{[EF]}_{AD})} \tag{22}$$

We recognise, $M_{ab} = -M_{ba}$ $(a,b = 0, 1, 2, 3)$ as the six generators of the Lorentz subgroup and decompose eqn. (21) into,

$$\underline{[M_{ab}, M_{cd}] = i(\eta_{ac} M_{bd} + \eta_{bd} M_{ac} - \eta_{ad} M_{bc} - \eta_{bc} M_{ad})} \tag{23a}$$

$$\underline{[M_{ab}, M_{5d}] = i(\eta_{ad} M_{5b} - \eta_{bd} M_{5a})} \tag{23b}$$

$$\underline{[M_{5b}, M_{5d}] = i M_{bd}} \tag{23c}$$

<u>The Dirac 4-Component Spinor Representation of SO(2,3)</u> is defined in direct analogy with the Lorentz spinors of Appendix A. Dirac $SO(2,3)$ matrices, Γ_A are required to satisfy,

$$\underline{\{\Gamma_A, \Gamma_B\} = 2\ \eta_{AB}\ I} . \tag{24}$$

This relation may be satisfied by 4×4 matrices $(\Gamma_A)_\alpha{}^\beta$ $(\alpha, \beta = 1,2,3,4)$ so that we may identify, from (24),

$\{\Gamma_a, \Gamma_b\} = \{\gamma_a, \gamma_b\}$ together with $\Gamma_5{}^2 = 1$ and $\{\Gamma_a, \Gamma_5\} = 0$. A suitable choice for the Γ_A is,

$$\underline{\Gamma_a = \gamma_a \gamma_5 \quad \text{and} \quad \Gamma_5 = i\gamma_5} \quad . \tag{25}$$

An alternative choice would be $\Gamma_a = \gamma_a$ and $\Gamma_5 = i\gamma_5$ but we shall also require Γ_A to satisfy

$$\underline{\Gamma_A{}^T = C\, \Gamma_A\, C^{-1}} \tag{26}$$

and this relation is only satisfied by (25).

($C = -C^T$ is the charge conjugation matrix, Appendix B).

With Γ_A given by (25) we may verify directly that the 10 matrices $\underline{\Sigma_{AB} = \frac{i}{2}[\Gamma_A, \Gamma_B]}$ span the 4×4 matrix representation of the algebra, (21) with,

$$\underline{(M_{AB})_\alpha{}^\beta = -\tfrac{1}{2}(\Sigma_{AB})_\alpha{}^\beta} \quad .$$

Decomposing with respect to the Lorentz subgroup we see that,

$$\underline{\Sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b] \equiv \sigma_{ab}} \quad \text{and} \quad \underline{\Sigma_{5b} = -\gamma_b} \quad . \tag{27}$$

Notice also that the Σ_{AB} satisfy,

$$\underline{\Sigma_{AB}{}^T = -\, C\, \Sigma_{AB}\, C^{-1}} \quad . \tag{28}$$

Four component Dirac $SO(2,3)$ spinors transform under $S \in SO(2,3)$ according to

$$\psi_\alpha \xrightarrow{S} \psi'_\alpha = \exp\left[-\frac{i}{4}\omega^{AB}\Sigma_{AB}\right]_\alpha^{\ \beta}\psi_\beta \tag{29a}$$

or, infinitesimally,

$$\delta\psi_\alpha = -\frac{i}{4}\omega^{AB}(\Sigma_{AB})_\alpha^{\ \beta}\psi_\beta \quad . \tag{29b}$$

The 16 matrices, I, Γ^A, Σ^{AB} form a complete set and hence, a basis for the expansion of any 4×4 matrix, so that we may rewrite the Fierz resummation of Appendix B as,

$$\phi_\alpha \overline{\psi}^\beta = \frac{1}{4}\{-(\overline{\phi}\,\psi)\delta_\alpha^{\ \beta} - \overline{\phi}\Gamma^A\psi(\Gamma_A)_\alpha^{\ \beta} + \tfrac{1}{2}\overline{\phi}\,\Sigma^{AB}\psi(\Sigma_{AB})_\alpha^{\ \alpha}\} \tag{30}$$

where ϕ_α and ψ_α are Dirac SO(2,3) spinors and $\overline{\psi} \equiv \psi^T C$.

The minimal grading of the algebra of SO(2,3) is achieved by adding the 4 'odd' generators, Q_α (α= 1,2,3,4), which together form a SO(2,3) spinor,

$$[M_{AB}, Q_\alpha] = \tfrac{1}{2}(\Sigma_{AB})_\alpha^{\ \beta}Q_\beta \tag{31a}$$

The GLA is closed with the anticommutator,

$$\{Q_\alpha, Q_\beta\} = -\tfrac{1}{2}(\Sigma^{AB}C)_{\alpha\beta}M_{AB} \quad . \tag{31b}$$

Introducing Grassmann parameters ε_α we may write (31b) as,

$$[\overline{\varepsilon}_1 Q, \overline{\varepsilon}_2 Q] = \tfrac{1}{2}\overline{\varepsilon}_1\Sigma^{AB}\varepsilon_2 M_{AB} \quad . \tag{31c}$$

Notice that the Q_α generators close on the Lorentz generators, M_{ab}, in eqn. (31b). In the Wess-Zumino group algebra the odd generators,

$\dot{S}_\alpha$ close only on translations, eqn. (12f). This is consistent in the expanded algebra when $[P_a, S_\alpha] \subset S$ and $[P_a, P_b] \subset M_{ab}$. (Our choice of the constants in (31b) is justified when we explicitly construct the algebra of OSp(1,4)).

The Homomorphism $Sp(4, \mathbb{R}) \overset{2:1}{\cong} SO(2,3)$

In Appendix B it is pointed out how the matrix C may be considered as a 4×4 symplectic spinorial metric, $C_{\alpha\beta} = - C_{\beta\alpha}$ in the vector space of Dirac spinors. Our interest is restricted to Majorana, self-conjugate spinors and we shall therefore find it most convenient to work in the Majora representation, in which the Γ_A and Σ_{AB} are all pure imaginary, (see Appendix B). In this representation,

$$C = i\left(\begin{array}{cc|cc} 0 & +1 & & \\ -1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & -1 \\ \multicolumn{2}{c|}{0} & +1 & 0 \end{array}\right) \equiv i\left(\begin{array}{c|c} -\varepsilon & 0 \\ \hline 0 & \varepsilon \end{array}\right)$$

$\therefore$ $C^2 = +1$ or $C^{-1} = C$ and we may use C to raise and lower Dirac spinor indices,

$$(\Gamma_A)_{\alpha\beta} \equiv (\Gamma_A)_\alpha{}^\gamma C_{\gamma\beta} = (\Gamma_A C)_{\alpha\beta}$$

$$\psi^\alpha \equiv \psi_\beta{}^T (C^{-1})^{\beta\alpha} = \psi_\beta{}^T C^{\beta\alpha} = \bar{\psi}^\alpha \quad \text{etc.}$$

$Sp(4, \mathbb{R})$ is the group of transformations which leaves invariant the symplectic quadratic form,

$$\underline{\bar{\psi}\,\psi = C^{\alpha\beta}\,\psi_\alpha\psi_\beta} \qquad (\text{where } \{\psi_\alpha, \psi_\beta\} = 0)$$

Let $A \in$ algebra $Sp(4, \mathbb{R})$ then we require A to satisfy

$$\delta_A(\bar{\psi}\,\psi) = \delta_A(\psi^T C\,\psi) = \psi^T(A^T C + CA)\psi = 0$$

i.e. $\underline{A^T = -\,C\,A\,C^{-1}}$

Comparing this relation with (28) and using the completeness of the set $\{I,\ \Gamma^A,\ \Sigma^{AB}\}$ we see that 'A' must be of the form

$$A_\alpha{}^\beta = \frac{i}{2}\,\omega^{AB}(\Sigma_{AB})_\alpha{}^\beta \qquad (\omega^{AB} \in \mathbb{R}) \;.$$

Since $-\frac{1}{2}(\Sigma_{AB})_\alpha{}^\beta$ form a 4×4 matrix representation of the SO(2,3) algebra, (21), we have demonstrated the <u>local isomorphism</u> between Sp(4, $\mathbb{R}$) and SO(2,3).

Globally there is a 2:1 homomorphism between Sp(4, $\mathbb{R}$) and SO(2,3), which contains the 2:1 homomorphism between SL(2,c) and SO(1,3) (Appendix A).

Formally this is written $\quad SO(2,3) = \dfrac{Sp(4,\ \mathbb{R})}{Z_2}$, but we shall henceforth identify the two groups.

The Group Osp(1,4)

OSp(m,2n)[32] is the group of transformations which leave invariant the quadratic form,

$$\underline{\delta_{ij}\ x_i x_j + C^{\alpha\beta}\ \psi_\alpha \psi_\beta \equiv \chi^2 + \bar{\psi}\,\psi}$$

where the x_i $(i = 1, \ldots, m)$ are Bose coordinates

ψ_α $(\alpha = 1, \ldots, 2n)$ are Fermi coordinates, $\{\psi_\alpha, \psi_\beta\} = 0$

and $C^{\alpha\beta} = -\,C^{\beta\alpha}$ is a symplectic metric.

The aggregate of all $V = (x_i, \psi_\alpha)^T$ constitutes an $m + 2n$ dimensional graded vector space[14], the fundamental representation space of Osp(m,2n).

We may write the quadratic form as, $V^T G V$, where

$$G = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & C \end{array}\right)$$ is the block diagonal metric for the graded vector space.

Now let $A \in \text{alg } OSp(m,2n)$ so that,

$$\underline{\delta_A V = A V}, \qquad \underline{A = \left(\begin{array}{c|c} p & q \\ \hline r & s \end{array}\right)}$$

with p - $m \times m$ Bose Square Matrix

s - $2n \times 2n$ Bose Square Matrix

r,q - $m \times 2n$, $2n \times m$ (respectively) Fermi Matrices.

By definition, 'A' must satisfy,

$$\delta_A (V^T G V) = 0 .$$

Now, $$\delta_A V^T = V^T \left(\begin{array}{c|c} p^T & r^T \\ \hline -q^T & s^T \end{array}\right)$$

so that,

$$\delta(V^T G V) = V^T \left[\left(\begin{array}{c|c} p^T & r^T \\ \hline -q^T & s^T \end{array}\right) \left(\begin{array}{c|c} I & 0 \\ \hline 0 & C \end{array}\right) + \left(\begin{array}{c|c} I & 0 \\ \hline 0 & C \end{array}\right) \left(\begin{array}{c|c} p & q \\ \hline r & s \end{array}\right) \right] V = 0$$

i.e. $\underline{P^T = -P}$ (i) $\Rightarrow$ $\underline{P \in \text{alg } O(m)}$

and

$\underline{r^T C = -q}$ (ii)

and

$\underline{q^T = Cr}$ (iii)

(ii), (iii) $\Rightarrow$ $\underline{q = -r^T C}$

and

$\underline{S^T C = -CS}$ (iv) $\Rightarrow$ $\underline{S \in \text{alg } Sp(2n)}$

Hence,

$$A = \left(\begin{array}{c|c} P_{ij} & -\bar{r}_i^{\ \alpha} \\ \hline r_{i\alpha} & S_\alpha^{\ \beta} \end{array}\right) \qquad (\bar{r} \equiv r^T C)$$

and P_{ij} and $S_\alpha^{\ \beta}$ are matrices in the fundamental representations of $O(m)$ and $Sp(2n)$ respectively.

Of relevance to supersymmetry and supergravity theories are the $O\tilde{s}p(m,4)$ groups, with $m \leqslant 8$.
The Bose sectors of these groups are,

$O(m)$ internal symmetry groups of extended $(m > 1)$ supersymmetry and supergravity theories, (see page 110).

$Sp(4) \cong SO(2,3)$ anti de Sitter group of space time symmetries.
The Fermi sector consists of the set of m spinorial supergauge translations parameterized by the $r_{i\alpha}$, $(\alpha = 1,2,3,4)$. Our interest is limited to minimal supergravity with $m = 1$. We therefore write an element $\underline{A \ \varepsilon \text{ alg } Osp(1,4)}$ as,

$$\underline{A = \left(\begin{array}{c|c} 0 & -\bar{\varepsilon} \\ \hline \varepsilon & S \end{array}\right)} \qquad (32)$$

where $\quad S_\alpha^{\ \beta} = -\frac{i}{4}\,\omega^{AB}(\Sigma_{AB})_\alpha^{\ \beta} \ \varepsilon \ Sp(4) \cong SO(2,3).$

We may identify the generators in this 5×5 fundamental representation by writing,

$$A = \frac{i}{2}\omega^{AB}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & -\frac{1}{2}\Sigma_{AB} \end{array}\right) + \bar{\varepsilon}^{\alpha}\left(\begin{array}{c|cccc} 0 & -\delta_\alpha^{\ 1} & -\delta_\alpha^{\ 2} & -\delta_\alpha^{\ 3} & -\delta_\alpha^{\ 4} \\ \hline C\alpha_1 & & & & \\ C\alpha_2 & & & & \\ C\alpha_3 & & 0 & & \\ C\alpha_4 & & & & \end{array}\right)$$

i.e. $\quad \underline{A = \frac{i}{2}\,\omega^{AB} M_{AB} + \bar{\varepsilon}^{\alpha} Q_\alpha}$.

The algebra of the generators $\{M_{AB}, Q_\alpha\}$ is then deduced from the commutator,

$$\left[A_1, A_2\right] = \left[\begin{pmatrix} 0 & -\bar{\varepsilon}_1 \\ \varepsilon_1 & -\frac{i}{4}\omega_1{}^{AB}\Sigma_{AB} \end{pmatrix}, \begin{pmatrix} 0 & \bar{\varepsilon}_2 \\ \varepsilon_2 & -\frac{i}{4}\omega_2{}^{AB}\Sigma_{AB} \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & -\bar{\varepsilon}_{12} \\ \varepsilon_{12} & -\frac{i}{4}\omega_{12}{}^{AB}\Sigma_{AB} \end{pmatrix}$$

where,

$$\underline{\varepsilon_{12} = -\frac{i}{4}(\omega_1{}^{AB}\Sigma_{AB}\,\varepsilon_2 - \omega_2{}^{AB}\Sigma_{AB}\,\varepsilon_1)} \tag{33a}$$

and

$$\underline{\omega_{12}{}^{AB} = -(\omega_1{}^{AC}\omega_2{}^{B}{}_{C} - \omega_2{}^{AC}\omega_1{}^{B}{}_{C}) - i\,\bar{\varepsilon}_1\,\Sigma^{AB}\,\varepsilon_2} \tag{33b}$$

These closure relations then imply, in particular that

$$\underline{\left[\frac{i}{2}\,\omega_1{}^{AB}\,M_{AB},\ \bar{\varepsilon}_2{}^{\alpha}Q_\alpha\right] = \frac{i}{4}(\bar{\varepsilon}_2\omega_1{}^{AB}\Sigma_{AB})^{\alpha}Q_\alpha}$$

and

$$\underline{\left[\bar{\varepsilon}^{\alpha}Q_\alpha\ \ \bar{\varepsilon}^{\beta}Q_\beta\right] = \frac{i}{2}(-i\,\bar{\varepsilon}_1\,\Sigma^{AB}\varepsilon_2)M_{AB} = \frac{1}{2}\,\bar{\varepsilon}_1\,\Sigma^{AB}\,\varepsilon_2\,M_{AB}}\,.$$

These relations may be identified as the relations (31) so that we have established that O$p(1,4) is the minimal grading of the anti de Sitter group.

Group Contraction OSp(1,4) $\longrightarrow$ Wess-Zumino Group

The parameter which characterizes the group contraction, $SO(2,3) \rightarrow ISO(1,3)$, is the radius, R, of the anti de Sitter space, $\eta^{AB}x_A x_B = R^2$. This space is an orbit of SO(2,3) in the 5-dimensional pseudo-Euclidean space with metric η_{AB} and is therefore isomorphic to the coset space, $\frac{SO(2,3)}{SO(1,3)}$, (see Appendix D). Group contraction then amounts to letting $R \rightarrow \infty$ so that the anti de Sitter space becomes a Minkowski space, $\frac{ISO(1,3)}{SO(1,3)}$,

$$\frac{SO(2,3)}{SO(1,3)} \xrightarrow{R \rightarrow \infty} \frac{ISO(1,3)}{SO(1,3)} \quad .$$

Clearly, group contraction doesn't involve the Lorentz, SO(1,3) subgroup. The coset generators, M_{5a}, must become translation generators, P_a, after group contraction and this is achieved through setting,

$$\underline{P_a = \frac{1}{R} M_{5a}} \quad . \tag{34a}$$

Similarly, the supergauge generators, Q_α, of Osp(1,4) which close on rotations (eqn. (31b)), must, after group contraction, close on translations (eqn. (12f)). This is achieved by setting,

$$S_\alpha = \frac{1}{\sqrt{R}} Q_\alpha \quad . \tag{34b}$$

Collecting together the commutation relations, (23) and (31), and substituting for P_a and S_α, the algebra of Osp(1,4) is written,

$$\underline{[M_{ab}, M_{cd}] = i(\eta_{ac}M_{bd} + \eta_{bd}M_{ac} - \eta_{ad}M_{bc} - \eta_{bc}M_{ad})} \tag{35a}$$

$$\underline{[M_{ab}, P_d] = i(\eta_{ad}P_b - \eta_{bd}P_a)} \tag{35b}$$

$$\underline{[P_b, P_d] = \frac{i}{R^2} M_{bd}} \tag{35c}$$

$$[M_{ab}, S_\alpha] = \tfrac{1}{2}(\sigma_{ab})_\alpha{}^\beta S_\beta \tag{35d}$$

$$[P_a, S_\alpha] = -\frac{1}{2R}(\gamma_a)_\alpha{}^\beta S_\beta \tag{35e}$$

$$\{S_\alpha, S_\beta\} = (\gamma^a C)_{\alpha\beta} P_a - \frac{1}{2R}(\sigma^{ab} C) M_{ab} \quad . \tag{35f}$$

As $R \to \infty$, this algebra contracts to the Wess-Zumino algebra, eqns. (12).

Representations of Osp(1,4)

We defined Osp(1,4) through its 5-dimensional (fundamental) vector representation,

$$\delta_A V = AV \qquad\qquad A = \left(\begin{array}{c|c} 0 & -\bar{\varepsilon} \\ \hline \varepsilon & -\frac{i}{4}\omega^{AB}\Sigma_{AB} \end{array}\right)$$

and

$$V = \begin{pmatrix} \phi \\ \psi_\alpha \end{pmatrix} \equiv \begin{pmatrix} \phi \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \qquad \begin{array}{ll} \phi & \text{is a Bose variable} \\ \psi_\alpha & \text{are Fermi variables.} \end{array}$$

In terms of components this transformation reads,

$$\delta_A \phi = -\bar{\varepsilon}\psi \tag{36a}$$

and

$$\delta_A \psi = \phi\varepsilon - \frac{i}{4}\omega^{AB}\Sigma_{AB}\psi \quad . \tag{36b}$$

Hence ϕ is an SO(2,3) scalar and ψ_α is a (Majorana) spinor. If ϕ and ψ are field variables in space-time then (36) describe supersymmetry transformations between Bose and Fermi fields. The SO(2,3) part corresponds to internal (spinorial) transformations and for local fields could be supplemented by the space-time generators,

$$M_{AB} = -i(x_A \partial_B - x_B \partial_A) \quad .$$

$$\text{i.e.} \quad (-\tfrac{1}{2}\Sigma_{AB})_\alpha{}^\beta \longrightarrow (-\tfrac{1}{2}\Sigma_{AB})_\alpha{}^\beta - \delta_\alpha{}^\beta\, i(x_A \partial_B - x_B \partial_A) \quad .$$

Then we would find, using the closure relations, (31c),

$$[\delta_{\varepsilon_1} \delta_{\varepsilon_2}]\phi(x) = \tfrac{1}{2}\bar{\varepsilon}_1 \Sigma^{AB} \varepsilon_2\, i(x_A\partial_B - x_B\partial_A)\phi(x) \quad .$$

To compare this with the closure relations, (2.18) for the Wess-Zumino supersymmetry of Chapter II, we must contract the algebra and hence define the Wess-Zumino supergauge parameter ε_α by

$$\underline{\varepsilon_\alpha \longrightarrow \lim_{R\to\infty} \sqrt{R}\,\varepsilon_\alpha} \qquad (\text{then } [\varepsilon_\alpha] = (\text{mass})^{-1/2}).$$

Space-time is identified as an anti de Sitter space of 'radius' R and, at a finite distance from the point (0, 0, 0, 0, R) we see that as $R \to \infty$, $\frac{M_{ab}}{R} \to 0$ and $\frac{M_{5a}}{R} \to i\,\partial_a$, so that

$$[\delta_{\varepsilon_1} \delta_{\varepsilon_2}]\phi(x) \xrightarrow{R\to\infty} \bar{\varepsilon}_1 \gamma^a \varepsilon_2\, i\,\partial_a \phi(x) \quad .$$

This is the form of the supersymmetry closure relations of Chapter II. Equations (36) are not space-time transformations in this sense. They are purely internal transformations between a scalar and a spinor field and so require them to be of the same dimension, $[\phi] = [\psi]$. We shall return to this point in the next chapter.

Tensor Representations are defined in the usual manner as the direct product of n copies of the fundamental, vector representation.

A second rank Osp(1,4) tensor transforms like, $U\overline{V} \equiv U_i V^j$ $(i,j = 1, \ldots, 5)$ where,

$$V_1\overline{V}_2 \equiv \begin{pmatrix}\phi_1 \\ \psi_1\end{pmatrix} (\phi_2 \ \overline{\psi}_2) = \left(\begin{array}{c|c}\phi_1\phi_2 & \phi_1\overline{\psi}_2 \\ \hline \psi_1\phi_2 & \psi_1\overline{\psi}_2\end{array}\right) \equiv M \equiv \left(\begin{array}{c|c} t & \overline{\mu} \\ \hline \nu & T\end{array}\right)$$

$$\therefore \quad \underline{\delta_A M = [A,M]} \ . \tag{37}$$

M constitutes a 25-dimensional reducible representation of Osp(1,4). It contains an invariant, graded trace, Trg M which is the sum of the diagonal, even-even elements minus the sum of the diagonal odd-odd elements,

$$\text{Trg } M = \phi_1\phi_2 - \psi_{1\alpha}\overline{\psi}_2^{\alpha} = \phi_1\phi_2 + \overline{\psi}_2\psi_1 \ .$$

This is clearly invariant by definition of the group Osp(1,4).
M also contains;

(i) 14-dimensional antisymmetric tensor representation, with $(MG)^T = -MG$

(ii) 10-dimensional (traceless) symmetric representation, with $(MG)^T = MG$

where G is the block diagonal metric, $G = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & C\end{array}\right)$.

Now, $MG = \left(\begin{array}{c|c} t & \overline{\mu}C \\ \hline \nu & TC\end{array}\right) \quad \therefore \quad (MG)^T = \left(\begin{array}{c|c} t & \nu^T \\ \hline \mu & (TC)^T\end{array}\right)$

and we recognise that the 14-dimensional representation is the adjoint representation of Osp(1,4) with values in the Lie algebra,

$$M_{antisym'} = \left(\begin{array}{c|c} 0 & -\bar{\mu} \\ \hline \mu & -\frac{i}{4} T^{AB}\Sigma_{AB} \end{array}\right)$$

μ - SO(2,3) Majorana spinor

T^{AB} - SO(2,3) antisymmetric tensor.

Similarly we see that the 10-dimensional symmetric representation may be written as, (using (26))

$$M_{sym'c} = \left(\begin{array}{c|c} 4\phi & \bar{\nu} \\ \hline \nu & \phi + y^A \Gamma_A \end{array}\right)$$

ϕ - SO(2,3) scalar

ν - SO(2,3) Majorana spinor

y^A - SO(2,3) vector

and $\text{Trg}\, M_{sym'c} = 0$ as required.

Higher rank irreducible tensor representations may be defined but we shall not require such objects in later work.

4) Gauge Theory of Osp(1,4), Spontaneously Broken to SL(2,C)

The gauge potentials of a standard Yang-Mills gauge theory of Osp(1,4) are contained in the connection form, $\omega \equiv \omega_\mu dx^\mu$, where

$$\omega_\mu = \left(\begin{array}{c|c} 0 & -\bar{\psi}_\mu \\ \hline \psi_\mu & -\frac{i}{4} B_\mu^{\ AB} \Sigma_{AB} \end{array}\right) .$$

The transformation of ω_μ under $A \in$ alg Osp(1,4) is (eqn. (8)),

$$\underline{\delta_A \omega_\mu = \partial_\mu A + [A, \omega_\mu]} \quad . \tag{38}$$

This is the infinitesimal form of the transformation of ω_μ under $\Omega \in$ Osp(1,4),

$$\underline{\omega_\mu \to \omega_\mu' = \Omega\, \omega_\mu\, \Omega^{-1} - \Omega\, \partial_\mu\, \Omega^{-1}}$$

where $\underline{\Omega = \exp(A)}$.

Now, $$A(x) = \left(\begin{array}{c|c} 0 & -\bar{\varepsilon}(x) \\ \hline \varepsilon(x) & -\frac{i}{4}\,\omega^{AB}(x)\Sigma_{AB} \end{array}\right)$$

and, since $\omega_\mu \subset \text{alg Osp}(1,4)$ we use the closure relation, (33), to evaluate the commutator in (38) and obtain,

$$\delta_A B_\mu^{\ AB} = \partial_\mu \omega^{AB} - (\omega^{AC} B_{\mu\ C}^{\ \ B} - B_\mu^{\ AC}\,\omega^B_{\ C}) - i\,\bar{\varepsilon}\,\Sigma^{AB}\,\psi_\mu$$

$$\delta_A \psi_\mu = \partial_\mu \varepsilon - \frac{i}{4}(\omega^{AB}\,\Sigma_{AB}\,\psi_\mu - B_\mu^{\ AB}\,\Sigma_{AB}\,\varepsilon) \quad .$$

Separating the Sl(2,C) content in these transformations we have,

$$\underline{\delta_A B_\mu^{\ ab} = D_\mu \omega^{ab} - (\omega^a\, B_\mu^{\ b} - \omega^b\, B_\mu^{\ a}) - i\,\bar{\varepsilon}\,\sigma^{ab}\,\psi_\mu} \tag{39a}$$

$$\underline{\delta_A B_\mu^{\ a} = D_\mu \omega^a + \omega^a_{\ b}\, B_\mu^{\ b} + i\,\bar{\varepsilon}\,\gamma^a\,\psi_\mu} \tag{39b}$$

$$\underline{\delta_A \psi_\mu = D_\mu \varepsilon - \frac{i}{2} B_\mu^{\ a}\,\gamma_a\,\varepsilon - \frac{i}{4}\,\omega^{ab}\sigma_{ab}\,\psi_\mu + \frac{i}{2}\,\omega^a\,\gamma_a\,\psi_\mu} \tag{39c}$$

where $\omega^a \equiv \omega^{5a}$, $B_\mu^{\ a} \equiv B_\mu^{\ 5d}$ and D_μ is the Lorentz covariant derivative. The field strengths in this gauge theory are defined by eqns. (9),

$$F_{\mu\nu} = \partial_\mu\,\omega_\nu - \partial_\nu\,\omega_\mu - [\omega_\mu, \omega_\nu] \quad .$$

Thus, writing, $$F_{\mu\nu} \equiv \left(\begin{array}{c|c} 0 & -\bar{\psi}_{\mu\nu} \\ \hline \psi_{\mu\nu} & -\frac{i}{4}\,B_{\mu\nu}^{\ \ AB}\,\Sigma_{AB} \end{array}\right)$$

we have

$$B_{\mu\nu}^{\ \ AB} = \partial_\mu\,B_\nu^{\ AB} - \partial_\nu\,B_\mu^{\ AB} + (B_\mu^{\ AC} B_{\nu\ C}^{\ \ B} - B_\nu^{\ AC}\,B_{\mu\ C}^{\ \ B}) + i\,\bar{\psi}_\mu\,\Sigma^{AB}\,\psi_\nu$$

$$\psi_{\mu\nu} = \partial_\mu \psi_\nu - \partial_\nu \psi_\mu + \frac{i}{4}(B_\mu{}^{AB} \Sigma_{AB} \psi_\nu - B_\nu{}^{AB} \Sigma_{AB} \psi_\mu) \quad .$$

Again, separating the SL(2,c) content these relations become,

$$\underline{B_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} + (B_\mu{}^a B_\nu{}^b - B_\nu{}^a B_\mu{}^b) + i\,\bar{\psi}_\mu \sigma^{ab} \psi_\nu} \tag{40a}$$

$$\underline{B_{\mu\nu}{}^a = D_\mu B_\nu{}^a - D_\nu B_\mu{}^a - i\,\bar{\psi}_\mu \gamma^a \psi_\nu} \tag{40b}$$

$$\underline{\psi_{\mu\nu} = D_\mu \psi_\nu - D_\nu \psi_\mu - \frac{i}{2}(B_\mu{}^a \gamma_a \psi_\nu - B_\nu{}^a \gamma_a \psi_\mu)} \tag{40c}$$

where $R_{\mu\nu}{}^{ab} = \partial_\mu B_\nu{}^{ab} - \partial_\nu B_\mu{}^{ab} + (B_\mu{}^{ac} B_\nu{}^b{}_c - B_\nu{}^{ac} B_\mu{}^b{}_c)$ is the SL(2,c) curvature. (Recall from Chapter I that $R_{\mu\nu\kappa\lambda} = h_{\kappa a} h_{\lambda b} R_{\mu\nu}{}^{ab}$ is the Riemann curvature tensor).

The Bose sector $(\psi_\mu = 0)$ of this gauge theory is the anti de Sitter gauge theory which we discussed in Chapter I, (see also Appendix D). We found that the vierbein fields of the Einstein-Cartan theory emerged as the gauge potentials corresponding to the broken generators in the spontaneously broken gauge theory, $SO(2,3) \xrightarrow{SSB} SO(1,3)$. Space-time was then identified as a cross-section in the bundle $E(V_4, \frac{SO(2,3)}{SO(1,3)}, SO(2,3), P)$ associated to the anti de Sitter bundle $P(V_4, SO(2,3))$. The role of the vierbein fields was to solder each fibre $\{\frac{SO(2,3)}{SO(1,3)}\}_{\underset{\sim}{x}}$ of E to the space-time manifold, V_4, at each point x on the cross-section. To obtain the vierbein fields in the Osp(1,4) theory we must therefore spontaneously break Osp(1,4) down to a group not involving the M_{5a} generators. There are two such groups:

(i) <u>Osp(1,2c)</u>[(45),(54)], which is the minimal grading of SL(2,c) by 2-component complex Weyl spinor generators. (Note: $Sp(2,c) \cong SL(2,c)$). This group has precisely the algebra of Osp(1,4), with $M_{5a} = 0$. However, it is not a subgroup, (since the structure constants $f_{\alpha\beta}{}^{5a} \neq 0$) and so doesn't fit into the scheme of spontaneous symmetry breaking required to identify the vierbein.

(ii) <u>SL(2,C)</u> which, as a subgroup of Osp(1,4) with generators $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & -\frac{1}{2}\sigma_{ab} \end{array}\right)$ and thus leaving M_{5a} and Q_α as the eight broken generators, appears to provide the only satisfactory algebraic structure.

We are thus led to view supergravity as a gauge theory of Osp(1,4), spontaneously broken down to the Lorentz (covering) group SL(2,c). There are two basic methods of constructing spontaneously broken gauge theories;

(a) Explicit use of Higgs fields, some of which acquire a non-vanishing vacuum expectation value <u>either</u> by the imposition of group invariant constraints <u>or</u> by the choice of a suitable potential in the action.

(b) Direct construction of non-linear actions(13),(36), composed of fields which transform non-linearly under the full group, but linearly under the stability (unbroken) subgroup, (see Appendix D).

Both these methods have been applied to the Osp(1,4) gauge theory by previous workers, method (a) by Chamseddine(7),(8), method (b) by MacDowell and Mansouri(54), Chang and Mansouri(11) and by Gursey and Marchildon(36).

The purpose of the work in the remainder of this chapter is to clarify and compare the results of the two approaches. In particular, our interest centres on the choice of possible gauge actions which retain local supersymmetric invariance after Osp(1,4) is broken and which reproduce the supergravity action after the contraction of Osp(1,4) to the Wess-Zumino group. Upon various points, our results differ from those of ref. (8) due to disagreements in the evaluation of various covariant derivatives and, more importantly, in the rescaling of fields in order to carry out the group contraction procedure.

(a) Spontaneous Breakdown Osp(1,4) $\to$ SL(2,C), Induced by Constrained Higgs Fields

Symmetry breaking involves the eight coset generators, M_{5a}, Q_α and must therefore be triggered by a constrained multiplet of more than eight Higgs fields, (including four Bose (M_{5a}) and four Fermi (Q_α) degrees of freedom). We also recognise that the Bose (SO(2,3)) sector of the symmetry breaking is triggered by the vector multiplet $y^A(x)$ satisfying the SO(2,3) invariant constraint,

$$\underline{y^A(x)\; y_A(x) \quad = \quad R^2\;.} \tag{41}$$

(See also Appendix D for details of the SO(2,3) theory).
This 5-vector, $y^A(x)$ is contained in the 10-dimensional traceless symmetric representation of Osp(1,4) which also contains a Majorana spinor and therefore is large enough to form a suitable Higgs multiplet. Hence we take the real symmetric multiplet, H as the Higgs multiplet where,

$$H \quad = \quad \left(\begin{array}{c|c} 4\phi(x) & \overline{\lambda}(x) \\ \hline \lambda(x) & \phi(x) + i\Gamma^A y_A(x) \end{array}\right) \cdot \quad (\mathrm{Trg}\; H = 0) \tag{42}$$

Spontaneous symmetry breaking is then triggered by two group invariant constraints on the field in H, which prevents them all from attaining zero vacuum expectation values. These constraints define an eight dimensional hypersurface in the vector space of the fields in H, which is the orbit of Osp(1,4) in this space. Now H transforms under Osp(1,4) according to (37), $\delta_A H = [A, H]$, so that $Trg(H^n)$, for all integer $n > 1$, are group invariant constraints. However, since the orbit of Osp(1,4) in H-space is eight dimensional, (always equal to the dimension of the coset space of Osp(1,4) with respect to the stability subgroup SL(2,c)), it follows that only two of the constraints can be independent. We therefore take as group invariant constraints,

$$Trg\ H^2 = 12\phi^2 + 2\bar{\lambda}\lambda + 4y^A y_A = \text{constant},\ K_1$$

and

$$Trg\ H^3 = 60\phi^3 + 15\phi\bar{\lambda}\lambda + 12\phi y^A y_A + 3i\bar{\lambda}\Gamma^A\lambda y_A = \text{constant},\ K_2$$

Now, since y^A is an anti de Sitter 5-vector and $\Gamma_A = (\gamma_a\gamma_5, i\gamma_5)$, it follows that the matrix $y^A\Gamma_A$ forms pseudo-scalars when sandwiched between two spinors. Clearly then we require $\phi(x)$ to be a pseudo-scalar field and we see that under parity,

$$Trg(H^2) \xrightarrow{\text{parity}} Trg(H^2)$$

$$Trg(H^3) \xrightarrow{\text{parity}} -Trg(H^3) \quad .$$

Hence we set $K_2 = 0$ $(K_1 \to K)$ and constraint equations become,

$$Trg\ H^2 = 12\phi^2 + 2\bar{\lambda}\lambda + 4y^A y_A = K \tag{43a}$$

$$Trg\ H^3 = 60\phi^3 + 15\phi\bar{\lambda}\lambda + 12\phi y^A y_A + 3i\bar{\lambda}\Gamma^A\lambda y_A = 0 \tag{43b}$$

These equations may be solved for y_5 (Lorentz scalar) and ϕ (Lorentz pseudo-scalar). A solution automatically breaks the symmetry down to $SL(2,\mathbb{C})$ and we may anticipate an $SL(2,\mathbb{C})$ invariant, finite power series in λ, (since $\lambda^n \equiv 0$ for $n > 4$)

$$\underline{y_5 = y_5(y_a,\lambda_\alpha) = f_1(y_a) + f_2(y_a)\bar{\lambda}\lambda + f_3(y_a)(\bar{\lambda}\lambda)^2} \tag{44a}$$

$$\underline{\phi = \phi(y_a, \lambda_\alpha) = f_4(y_a)\bar{\lambda}\Gamma^5\lambda + y_a f_5(y_a)\bar{\lambda}\Gamma^a\lambda} \tag{44b}$$

where f_1, f_2, f_3, f_4, f_5 are Lorentz invariant functions of y_a. Substituting (44a) and (44b) into (43) and using

$$(\bar{\lambda}\lambda)(\bar{\lambda}\Gamma^A\lambda) \equiv 0 \quad \text{and} \quad \bar{\lambda}\Gamma^A\lambda\bar{\lambda}\Gamma^B\lambda = -\eta^{AB}(\bar{\lambda}\lambda)^2,$$

then equating the coefficients of the bilinears in λ yields the five functions, f_i, explicitly so that equations (44) become,

$$y_5 = \pm \frac{1}{\sqrt{\frac{K}{4} - y^a y_a}} \left[\left(\frac{K}{4} - y^a y_a\right) - \frac{\bar{\lambda}\lambda}{4} - \frac{(\bar{\lambda}\lambda)^2}{2K} \left(\frac{K - 3y^a y_a}{K - 4y^a y_a}\right) \right]$$

$$\phi = \pm \frac{i}{K} \sqrt{\frac{K}{4} - y^a y_a}\ \bar{\lambda}\Gamma^5\lambda + \frac{i}{K} y_a \bar{\lambda}\Gamma^a\lambda \quad .$$

If we look at the Bose sector $(\lambda_\alpha = 0)$ we see that these solutions become,

$$y_5 = \pm\sqrt{\frac{K}{4} - y^a y_a} \quad \text{and} \quad \phi = 0.$$

Hence, for a direct extension of the $SO(2,3)$ constraint, (41), we set $K = 4R^2$ and the solutions for y_5 and ϕ become,

$$y_5 = \frac{1}{\sqrt{R^2 - y^a y_a}} \left[(R^2 - y^a y_a) - \frac{1}{4}\bar{\lambda}\lambda - \frac{(\bar{\lambda}\lambda)^2}{32R^2} \frac{(4R^2 - 3y^a y_a)}{(R^2 - y^a y_a)} \right] \tag{45a}$$

and

$$\phi = \underline{\frac{i}{4R^2}\left[\sqrt{R^2 - y^a y_a}\;\bar{\lambda}\Gamma^5\lambda + y_a\;\bar{\lambda}\Gamma^a\lambda\right]} \quad . \tag{45b}$$

Now the component form of the transformation laws $\delta_A H = [A,H]$ are,

$$\delta_A\phi = -\frac{1}{2}\bar{\varepsilon}\lambda$$

$$\delta_A\lambda = 3\phi\varepsilon - iy^A\Gamma_A\varepsilon - \frac{i}{4}\omega^{AB}\Sigma_{AB}\lambda$$

and

$$\delta_A y_A = \frac{i}{2}\bar{\varepsilon}\Gamma_A\lambda + \omega_A{}^B y_B \quad .$$

Eliminating the fields y_5 and ϕ through the constraints (45) then gives

$$\delta_A\lambda = \frac{3i}{4R^2}\left[\sqrt{R^2 - y^a y_a}\;\bar{\lambda}\Gamma^5\lambda + y_a\bar{\lambda}\Gamma^a\lambda\right]\varepsilon - i\,y^a\Gamma_a\varepsilon$$

$$\frac{-i}{\sqrt{R^2 - y^a y_a}}\left[(R^2 - y^a y_a) - \frac{\bar{\lambda}\lambda}{4} - \frac{(\bar{\lambda}\lambda)^2}{32R^2}\left(\frac{4R^2 - 3y^a y_a}{R^2 - y^a y_a}\right)\right]\Gamma_5\varepsilon$$

$$- \frac{i}{4}\omega^{AB}\Sigma_{AB}\lambda$$

and

$$\delta_A y_a = \frac{i}{2}\bar{\varepsilon}\Gamma_a\lambda + \omega_a{}^b y_b$$

$$+ \frac{\omega_a{}^5}{\sqrt{R^2 - y^a y_a}}\left[(R^2 - y^a y_a) - \frac{\bar{\lambda}\lambda}{4} - \frac{(\bar{\lambda}\lambda)}{32R^2}\left(\frac{4R^2 - 3y^a y_a}{R^2 - y^a y_a}\right)\right] .$$

These rather complicated transformation laws are required in order that the 'coordinates' (y_a, λ_α) remain on the 8-dimensional hypersurface defined by $\mathrm{Trg}\,H^2 = 4R^2$ and $\mathrm{Trg}\,H^3 = 0$. The λ_α and y_a have inhomogeneous transformation laws under $Osp(1,4)$ and can never have an invariant vacuum expectation value. They are the Goldstone modes[28] in the theory, connecting equivalent vacua. In particular the choice $y_a = 0 = \lambda_\alpha$, of vacuum is lost after an infinitesimal Osp(1,4) transformation since,

$$\underline{\langle\delta\ \lambda\rangle_{vac} \equiv \delta_A\lambda\Big|_{y_a = \lambda_\alpha = 0} = -iR\Gamma_5\varepsilon} \tag{46a}$$

and

$$\underline{\langle\delta_A y_a\rangle_{vac} \equiv \delta_A y_a\Big|_{\lambda = y_a = 0} = +R\ \omega_a{}^5 = -R\ \omega_a} \tag{46b}$$

When the Osp(1,4) transformations are local with $A = A(x)$ then the inhomogeneous nature of the transformations, $\delta_A\lambda$ and $\delta_A y_a$ allows the field $\lambda_\alpha(x)$ and $y_a(x)$ to be removed by a suitable gauge choice. This gauge is called the unitary gauge, (see also Appendix D). In standard Yang-Mills gauge theories of internal phase symmetries it is in the unitary gauge where the Goldstone Bosons vanish and yet their physical degrees of freedom remain since in this gauge, the gauge bosons corresponding to the spontaneously broken generators acquire mass terms (Appendix D). In the SO(2,3) spontaneously broken gauge theory of gravity of Chapter I and Appendix D, gauging away the four 'Goldstone Modes', $y_a(x)$ gave rise to a cosmological term and no mass terms. We therefore cannot regard the $y_a(x)$ as possessing any physical degrees of freedom; they are auxiliary fields required to build a geometrical model. We shall adopt the same philosophy for the fields, $y_a(x)$ and $\lambda_\alpha(x)$ occurring in H. In the unitary gauge where $\underline{y_a(x) = \lambda_\alpha(x) = 0}$ we see from (45) that, $\underline{\phi = 0}$ and $\underline{y_5 = R}$

$$\underline{\therefore\ \underline{H}\Big|_u = -R\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \gamma_5 \end{array}\right)} \tag{47a}$$

where $\underline{\ \ }\Big|_u$ refers to a matrix evaluated in the unitary gauge. Now,

$$\nabla_\mu H = \partial_\mu H - [\omega_\mu , H] \equiv \left(\begin{array}{c|c} 4\nabla_\mu \phi & \overline{\nabla_\mu \lambda} \\ \hline \nabla_\mu \lambda & \nabla_\mu \phi + i\Gamma^A \nabla_\mu y_A \end{array}\right)$$

so that

$$\nabla_\mu H\Big]_u = R \left(\begin{array}{c|c} 0 & -\bar{\psi}_\mu \gamma_5 \\ \hline -\gamma_5 \psi_\mu & i\Gamma^a B_{\mu a} \end{array}\right) \tag{47b}$$

Dimensions of Fields

Before constructing Osp(1,4) invariants, as candidates for physical actions, it is helpful at this stage to consider more carefully the dimensions of the fields involved in the theory. First consider the gauge fields, which are the components of ω_μ. Since the field strengths $F_{\mu\nu}$ are defined from,

$$F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - [\omega_\mu , \omega_\nu] ,$$

we require

$$[\omega_\mu] = [\partial_\mu] = \text{mass}.$$

Clearly we do not want $[B_\mu{}^{ab}] = [B_\mu{}^a] = [\psi_{\mu\alpha}] = \text{mass}$, so that we must rescale $B_\mu{}^a$ and ψ_μ so that they have the dimensions of the vierbein and gravitino field respectively. The two dimensionful parameters which we have available are the gravitational coupling constant κ ($[\kappa] = (\text{mass})^{-1}$) and the group contraction parameter R ($[R] = (\text{mass})^{-1}$).

Now, since $$A = \left(\begin{array}{c|c} 0 & -\bar{\varepsilon} \\ \hline \varepsilon & -\frac{i}{4}\omega^{AB}\Sigma_{AB} \end{array}\right) \quad \varepsilon \text{ algebra of Osp(1,4)}$$

we require,

$$[A] = [\varepsilon_\alpha] = [\omega^{AB}] = 0 \quad (\text{since } [A\ A] \subset A) \ .$$

In our previous discussion of the group contraction procedure, $Osp(1,4) \to$ Wess-Zumino, we found it necessary to rescale the generators M_{5a} and Q_α according to (34), $M_{5a} \to P_a = \frac{1}{R} M_{5a}$ and $Q_\alpha \to S_\alpha = \frac{1}{\sqrt{R}} Q_\alpha$. For the matrix representation (32) of $Osp(1,4)$, $A = -\frac{i}{4}\omega^{ab}\sigma_{ab} + \frac{i}{2}\omega^a\gamma_a + \bar{\varepsilon}^\alpha Q_\alpha$ we may carry out the group contraction procedure by rescaling the parameters ω^a and ε_α according to

$$\underline{\omega^a \to \xi^a = R\omega^a} \qquad \text{and} \qquad \underline{\varepsilon_\alpha \to \varepsilon_\alpha = \sqrt{R}\ \varepsilon_\alpha} \tag{48}$$

$$\text{then} \quad A = -\frac{i}{4}\omega^{ab}\sigma_{ab} + \frac{i}{2}\xi^a \frac{1}{R}\gamma_a + \bar{\varepsilon}^\alpha \frac{1}{\sqrt{R}} Q_\alpha$$

$$\equiv -\frac{i}{4}\omega^{ab}\sigma_{ab} + \frac{i}{2}\xi^a P_a + \bar{\varepsilon}^\alpha S_\alpha \ .$$

We therefore rescale the parameters ω_a and ε_α according to (48) and write,

$$A = \left(\begin{array}{c|c} 0 & -\sqrt{m}\,\bar{\varepsilon} \\ \hline \sqrt{m}\,\varepsilon & -\frac{i}{4}\omega^{ab}\sigma_{ab} + \frac{im}{2}\xi^a\gamma_a \end{array}\right) \quad (m \equiv \tfrac{1}{R}) \tag{49}$$

and now,

$[\xi^a] = $ length - translation parameter

$[\varepsilon_\alpha] = (\text{length})^{\frac{1}{2}}$ - supergauge translation

$[\omega_{\alpha\beta}] = 0$ - Lorentz transformations.

If we rewrite the closure relations (33) in terms of the rescaled fields we have

$$[A_1,A_2] = A_{12} \equiv \left(\begin{array}{c|c} 0 & -\sqrt{m}\,\bar{\varepsilon}_{12} \\ \hline \sqrt{m}\,\varepsilon_{12} & -\frac{i}{4}\,\omega_{12}{}^{ab}\sigma_{ab} + \frac{im}{2}\,\xi_{12}^{a}\gamma_a \end{array}\right)$$

with

$$\underline{\omega_{12}{}^{ab} = -(\omega_1{}^{ac}\omega_2{}^{b}{}_{c} - \omega_2{}^{ac}\omega_1{}^{b}{}_{c}) - m^2(\xi_1{}^{a}\xi_2{}^{b} - \xi_2{}^{a}\xi_1{}^{b})}$$

$$\underline{-\,im\,\bar{\varepsilon}_1\,\sigma^{ab}\,\varepsilon_2} \tag{50a}$$

and

$$\underline{\xi_{12}{}^{a} = (\omega_1{}^{a}{}_{b}\,\xi_2{}^{b} - \omega_2{}^{a}{}_{b}\,\xi_1{}^{b}) + i\,\bar{\varepsilon}_1\,\gamma^{a}\,\varepsilon_2} \tag{50b}$$

and

$$\underline{\varepsilon_{12} = -\frac{i}{4}(\omega_1{}^{ab}\,\sigma_{ab}\,\varepsilon_2 - \omega_2{}^{ab}\,\sigma_{ab}\,\varepsilon_1) + \frac{im}{2}(\xi_1{}^{a}\,\gamma_a\,\varepsilon_2 - \xi_2{}^{a}\,\gamma_a\,\varepsilon_1)}\,. \tag{50c}$$

Upon group contraction, $m \to 0$ we see that the closure relations reduce to those of the Wess-Zumino group (equations (12)). A general problem with theories related by group contraction is that the relationship cannot be established stage by stage[(8),(77)]. In fact one must evaluate the 'larger' theory down to group scalars and only then let $R \to \infty$ ($m \to 0$). In the present case it is clear from eqn. (44) (or (34)) that we cannot carry out the group contraction procedure on the matrices, 'A', and must leave $m \neq 0$ until we have evaluated scalar quantities.

This rescaling of the parameters in the algebra of Osp(1,4) will also apply to any multiplet of fields in the adjoint representation. In particular the connection form ω_μ becomes,

$$\underline{\omega_\mu = \left(\begin{array}{c|c} 0 & -\sqrt{m}\,\bar{\psi}_\mu \\ \hline \sqrt{m}\,\psi_\mu & -\frac{i}{4}\,B_\mu{}^{ab}\sigma_{ab} + \frac{im}{2}\,h_\mu{}^{a}\,\gamma_a \end{array}\right)} \tag{51a}$$

where $\underline{\psi_\mu \to \sqrt{m}\,\psi_\mu}$ and $\underline{B_\mu{}^a \to h_\mu{}^a = m\,B_\mu{}^a}$.

Similarly the field strength tensor $F_{\mu\nu}$ becomes,

$$\underline{F_{\mu\nu} = \left(\begin{array}{c|c} 0 & -\sqrt{m}\,\bar{\psi}_{\mu\nu} \\ \hline \sqrt{m}\,\psi_{\mu\nu} & -\frac{i}{4} B_{\mu\nu}{}^{ab}\sigma_{ab} + \frac{im}{2} h_{\mu\nu}{}^a \gamma_a \end{array}\right)} \quad . \tag{51b}$$

We see that the component form of the Osp(1,4) infinitesimal gauge transformations, eqns. (39) are now

$$\underline{\delta_A B_\mu{}^{ab} = D_\mu \omega^{ab} - m^2(\xi^a h_\mu{}^b - \xi^b h_\mu{}^a) - im\,\bar{\varepsilon}\,\sigma^{ab}\,\psi_\mu} \tag{52a}$$

$$\underline{\delta_A h_\mu{}^a = D_\mu \xi^a + \omega^a{}_b h_\mu{}^b + i\,\bar{\varepsilon}\,\gamma^a\,\psi_\mu} \tag{52b}$$

$$\underline{\delta_A \psi_\mu = D_\mu \varepsilon - \frac{i}{4}\,\omega^{ab}\,\sigma_{ab}\,\psi_\mu - \frac{im}{2} h_\mu{}^a \gamma_a \varepsilon + \frac{im}{2}\,\xi^a \gamma_a \psi_\mu} \tag{52c}$$

Clearly with $m = 0$, these eqns. (52) reduce to the transformations (16) of the Wess-Zumino gauge fields. Similarly the components of the field strength tensor are, (eqns. (40))

$$\underline{B_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} + m^2(h_\mu{}^a h_\nu{}^b - h_\nu{}^a h_\mu{}^b) + im\,\bar{\psi}_\mu \sigma^{ab} \psi_\nu} \tag{53a}$$

$$\underline{h_{\mu\nu}{}^a = D_\mu h_\nu{}^a - D_\nu h_\mu{}^a - i\,\bar{\psi}_\mu\,\gamma^a \psi_\nu} \tag{53b}$$

$$\underline{\psi_{\mu\nu} = D_\mu \psi_\nu - D_\nu \psi_\mu - \frac{im}{2}(h_\mu{}^a \gamma_a \psi_\nu - h_\nu{}^a\,\gamma_a \psi_\mu)} \, . \tag{53c}$$

As $m \to 0$ these become the Wess-Zumino field strengths, (18). With the rescaled fields in ω_μ we see that,

$[B_\mu{}^{ab}]$ = (mass) and we identify $B_\mu{}^{ab}$ as the spin connection components.

$[h_\mu{}^a] = 0$ and, from our analysis in Chapter I and Appendix B, we identify $h_\mu{}^a$ as the vierbein fields.

$[\psi_\mu] = (\text{mass})^{\frac{1}{2}}$, in order that we may identify ψ_μ as the gravitino field we must rescale it by the gravitational coupling constant κ, $\psi_\mu \to \kappa\,\psi_\mu$ so that $[\psi_\mu] \to (\text{mass})^{3/2}$.

We may leave the rescaling of ψ_μ by κ until a later stage.

The problem of the dimensionalities of fields other than those in the adjoint multiplet is left until the next chapter. We simply remark here that we shall always introduce the Higgs multiplet H in the dimensionless combination, $\frac{1}{R}H = mH$.

Osp(1,4) Invariant Actions

It is not difficult to write down Osp(1,4) invariants, constructed from the covariants, H, $\nabla_\mu H$ and $F_{\mu\nu}$. Examples of such invariants include, $\mathrm{Trg}(H^n)$, $\mathrm{Trg}(H^n F_{\mu\nu})$, $\mathrm{Trg}(H^n\ \nabla_\mu H)$ etc. However, for physical actions we also require scalar densities, invariant under general coordinate transformations, $x^\mu \to x'^\mu\,(x^\nu)$. As was the case for the SO(2,3) theory, we cannot use the metric tensor, $g_{\mu\nu}$ to contract world indices since $g_{\mu\nu}$ is given by, $g_{\mu\nu} = h_\mu{}^a h_\nu{}^b \eta_{ab}$ which, in view of (52b) is clearly not an Osp(1,4) invariant. All space-time invariants must therefore be constructed as differential 4-forms, the indices contracted with $\varepsilon^{\mu\nu\kappa\lambda}$, (see Appendix C).

We are therefore restricted to the following classes of Lagrangian densities, all of dimension $(\text{mass})^4$,

(a) $\underline{\mathcal{L}_{(1)} \sim \varepsilon^{\mu\nu\kappa\lambda}\ \mathrm{Trg}(mH)^n\ F_{\mu\nu}F_{\kappa\lambda}}$ (54a)

(b) $\underline{\mathcal{L}_{(2)} \sim \varepsilon^{\mu\nu\kappa\lambda}\ \mathrm{Trg}(mH)^n\ \nabla_\mu mH\ \nabla_\nu mH\ F_{\kappa\lambda}}$ (54b)

(c) $\underline{\mathcal{L}_{(3)} \sim \varepsilon^{\mu\nu\kappa\lambda}\ \mathrm{Trg}(mH)^n\ \nabla_\mu mH\ \nabla_\nu mH\ \nabla_\kappa mH\ \nabla_\lambda mH}$ (54c)

where n is an integer $\geqslant 0$.

The Osp(1,4) invariance of these Lagrangians is spontaneously broken to the Lorentz symmetry, SL(2,c) by the imposition of the group invariant constraints, (43). These constraints may be included into the Lagrangian through the terms,

$$\ell_1(x)\left[\mathrm{Trg}\ H^2 - 4R^2\right] + \ell_2(x)\ \mathrm{Trg}\ H^3.$$

The fields in H and the scalars, $\ell_1(x)$ and $\ell_2(x)$ are all auxiliary fields, the field equations for $\ell_1(x)$ and $\ell_2(x)$ being the constraints (43), (cf. SO(2,3) model[77] in Appendix D). The explicit form of the actions (54), in a general gauge where the fields $\lambda\alpha(x)$ and $y_a(x)$ are present, is very complicated[8]. We shall restrict our analysis to the unitary gauge where, $mH = -\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \gamma_5 \end{array}\right)$, so that $(mH)^{2n} = \pm 1$ and $(mH)^{2n+1} = \pm\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \gamma_5 \end{array}\right)$. Clearly we need therefore only consider $n = 0, 1$ in equations (54).

Gauge Actions in the Unitary Gauge

Let us consider first, the Lagrangians, $\mathcal{L}_{(1)}$, quadratic in the gauge field strengths $F_{\mu\nu}$. Since we require an action with a spontaneously broken symmetry we cannot take $n = 0$ in $\mathcal{L}_{(1)}$ and we are therefore led to consider,

$$\mathcal{L}_{(1)} = \epsilon^{\mu\nu\kappa\lambda} \operatorname{Trg} mH F_{\mu\nu} F_{\kappa\lambda} .$$

In the unitary gauge we see that,

$$\mathcal{L}_{(1)}\Big]_u = \epsilon^{\mu\nu\kappa\lambda} \operatorname{Trg}\left[- \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \gamma_5 \end{array}\right) F_{\mu\nu} F_{\kappa\lambda}\right]$$

$$= -\epsilon^{\mu\nu\kappa\lambda} \operatorname{Tr}(m\gamma_5 \psi_{\mu\nu}\overline{\psi}_{\kappa\lambda} + \frac{1}{16} B_{\mu\nu}{}^{AB} B_{\kappa\lambda}{}^{CD} \gamma_5 \Sigma_{AB} \Sigma_{CD})$$

i.e. $$\mathcal{L}_{(1)}\Big]_u = m\epsilon^{\mu\nu\kappa\lambda}\overline{\psi}_{\mu\nu}\gamma_5\psi_{\kappa\lambda} - \frac{1}{4}\epsilon^{\mu\nu\kappa\lambda} \epsilon_{abcd} B_{\mu\nu}{}^{ab} B_{\kappa\lambda}{}^{cd} .$$

Substituting the expressions, (53), for the field strengths we find that,

$$m\,\epsilon^{\mu\nu\kappa\lambda} \overline{\psi}_{\mu\nu}\gamma_5 \psi_{\kappa\lambda} = 4m\epsilon^{\mu\nu\kappa\lambda}\,\overline{D_\mu\psi_\nu}\,\gamma_5 D_\kappa\psi_\lambda +$$

$$+ 4im^2 \epsilon^{\mu\nu\kappa\lambda}\overline{\psi}_\mu \gamma_5\gamma_\nu D_\kappa\psi_\lambda + m^3 \epsilon^{\mu\nu\kappa\lambda}\overline{\psi}_\mu \gamma_5\gamma_\nu\gamma_\kappa\psi_\lambda$$

and

$$-\frac{1}{4}\epsilon^{\mu\nu\kappa\lambda}\epsilon_{abcd} B_{\mu\nu}{}^{ab} B_{\kappa\lambda}{}^{cd} = \epsilon^{\mu\nu\kappa\lambda}\epsilon_{abcd}\Big[-\frac{1}{4} R_{\mu\nu}{}^{ab} R_{\kappa\lambda}{}^{cd} +$$

$$- m^4 h_\mu{}^a h_\nu{}^b h_\kappa{}^c h_\lambda{}^d - \frac{im}{2}\overline{\psi}_\mu \sigma^{ab}\psi_\nu R_{\kappa\lambda}{}^{cd}$$

$$- m^2 h_\mu{}^a h_\nu{}^b R_{\kappa\lambda}{}^{cd} - im^3 h_\mu{}^a h_\nu{}^b \overline{\psi}_\kappa \sigma^{cd}\psi_\lambda + \frac{m^2}{4}\overline{\psi}_\mu\sigma^{ab}\psi_\nu\overline{\psi}_\kappa\sigma^{cd}\psi_\lambda\Big] .$$

However, a Fierz reshuffle shows that,

$$\underline{\epsilon^{\mu\nu\kappa\lambda}\epsilon_{abcd}\overline{\psi}_\mu\sigma^{ab}\psi_\nu\overline{\psi}_\kappa\sigma^{cd}\psi_\lambda \equiv 0} \qquad (55)$$

so that we may drop this term and combine the remaining terms in $\mathcal{L}_{(1)}\Big]_u$ in powers of 'm' to obtain,

$$\underline{\mathcal{L}_{(1)}\Big|_u = -\frac{1}{4}\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,R_{\mu\nu}{}^{ab}\,R_{\kappa\lambda}{}^{cd}}$$

$$\underline{+\,m\{4\varepsilon^{\mu\nu\kappa\lambda}\,\overline{D_\mu\psi_\nu}\,\gamma_5 D_\kappa\psi_\lambda - \frac{i}{2}\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,\bar{\psi}_\mu\,\sigma^{\alpha\beta}\psi_\nu\,R_{\kappa\lambda}{}^{cd}\}}$$

$$\underline{+\,m^2\{-\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,h_\mu{}^a h_\nu{}^b R_{\kappa\lambda}{}^{cd} + 4i\,\varepsilon^{\mu\nu\kappa\lambda}\,\bar{\psi}_\mu\,\gamma_5\gamma_\nu D_\kappa\psi_\lambda\}}$$

$$\underline{+\,m^3\{-3i\,\varepsilon^{\mu\mu\kappa\lambda}\,\bar{\psi}_\mu\gamma_5\sigma_{\nu\kappa}\psi_\lambda\}}$$

$$\underline{+\,m^4\{-\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,h_\mu{}^a h_\nu{}^b h_\kappa{}^c\,h_\lambda{}^d\}} \quad .$$

Now the Bose, Sp(4), part of the above action is simply,

$$\mathcal{L}_{(1)}\Big|_{u,\psi_\mu=0} = -\frac{1}{4}\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\Big[R_{\mu\nu}{}^{ab}\,R_{\kappa\lambda}{}^{cd} + \\ + 4m^2 h_\mu{}^a h_\nu{}^b R_{\kappa\lambda}{}^{cd} + 4m^4 h_\mu{}^a h_\nu{}^b h_\kappa{}^c\,h_\lambda{}^d\Big] \quad .$$

This is the gravity action[77] of Appendix D and following the analysis there, we multiply $\mathcal{L}_{(1)}\Big|_u$ by a dimensionless coupling constant, $\frac{1}{\Lambda^2}$ and identify $\frac{\Lambda}{m}$ as the gravitational coupling constant, κ,

$$\underline{\mathcal{L}_{(1)}\Big|_u = -\frac{1}{4m^2\kappa^2}\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,R_{\mu\nu}{}^{ab}\,R_{\kappa\lambda}{}^{cd}}$$

$$\underline{+\,\frac{1}{m\kappa^2}\{4\varepsilon^{\mu\nu\kappa\lambda}\,\overline{D_\mu\psi_\nu}\gamma_5 D_\kappa\psi_\lambda - \frac{i}{2}\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,\bar{\psi}_\mu\,\sigma^{ab}\psi_\nu\,R_{\kappa\lambda}{}^{cd}\}}$$

$$\underline{+\,\frac{1}{\kappa^2}\{-\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,h_\mu{}^a h_\nu{}^b\,R_{\kappa\lambda}{}^{cd} + 4i\varepsilon^{\mu\nu\kappa\lambda}\,\bar{\psi}_\mu\,\gamma_5\gamma_\nu D_\kappa\psi_\lambda\}}$$

$$\underline{-\,\frac{3im}{\kappa^2}\,\varepsilon^{\mu\nu\kappa\lambda}\,\bar{\psi}_\mu\,\gamma_5\,\sigma_{\nu\kappa}\psi_\lambda}$$

$$\underline{-\,\frac{m^2}{\kappa^2}\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,h_\mu{}^a\,h_\nu{}^b\,h_\kappa{}^c\,h_\lambda{}^d} \qquad (56)$$

The first term in $\mathcal{L}_{(1)}\Big|_u$ is the Gauss-Bonnet topological invariant, which also occurs in the SO(2,3) theory, and may be dropped. The two terms independent of m are those terms which occurred in the supergravity Lagrangian, $\mathcal{L}_{SG}$ of Chapter II equation (2.46). Notice however that the ratio of the terms is not the same so that they are not invariant under the local supersymmetry of the Wess-Zumino gauge theory. The two terms with the factor m^{-1} cause problems if we wish to carry out the group contraction procedure, $m \to 0$. We shall show later that a particular combination of these terms is a topological invariant and simply remark here that the combination in $\mathcal{L}_{(1)}\Big|_u$ does <u>not</u> produce a topological invariant.

The remaining terms are linear and quadratic in m and vanish upon group contraction. The quadratic term,

$$- \frac{m^2}{\kappa^2} \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{abcd} h_\mu{}^a h_\nu{}^b h_\kappa{}^c h_\lambda{}^d = -24 \frac{m^2}{\kappa^2} \det h_{\mu a}$$

contributes a cosmological term to the Einstein action with a cosmological constant, $-6m^2$ (as discussed in Appendix D). The linear term,

$$- \frac{3im}{\kappa^2} \varepsilon^{\mu\nu\kappa\lambda} \bar{\psi}_\mu \gamma_5 \sigma_{\nu\kappa} \psi_\lambda$$

is a mass term(17) for the gravitino field ψ_μ, with the mass being proportional to 'm', (Chamseddine(8) obtains a mass term $\sim \frac{1}{\kappa}$). Having identified the various terms in $\mathcal{L}_{(1)}\Big|_u$ we shall defer any further analysis until we have found the topological invariants and now proceed to evaluate $\mathcal{L}_{(2)}\Big|_u$ and $\mathcal{L}_{(3)}\Big|_u$, (equations (54)).

$$\mathcal{L}_{(2)}\Big|_u = \varepsilon^{\mu\nu\kappa\lambda} \operatorname{Trg}(mH\, m^2\, \nabla_\mu H\, \nabla_\nu H\, F_{\kappa\lambda})\Big|_u$$

Using equations (47) we find,

$$\mathcal{L}_{(2)}\Big|_u = \varepsilon^{\mu\nu\kappa\lambda}(-im^2 h_\mu{}^a \bar{\psi}_\nu \Gamma_a \psi_{\kappa\lambda} + \frac{im}{8}\varepsilon_{abcd}\bar{\psi}_\mu \sigma^{cd}\psi_\nu B_{\kappa\lambda}{}^{ab} +$$

$$+ m^2 h_\mu{}^a h_\nu{}^b B_{\kappa\lambda}{}^{cd}\varepsilon_{abcd}) \quad .$$

Again, substituting for the field strengths, (53) and multiplying by $\frac{1}{\Lambda^2}$,

$$\underline{\mathcal{L}_{(2)}\Big|_u = \frac{i}{8m\kappa^2}\varepsilon^{\mu\nu\kappa\lambda}\varepsilon_{abcd}\bar{\psi}_\mu \sigma^{ab}\psi_\nu R_{\kappa\lambda}{}^{cd}}$$

$$\underline{+ \frac{1}{\kappa^2}\{\varepsilon^{\mu\nu\kappa\lambda}\varepsilon_{abcd} h_\mu{}^a h_\nu{}^b R_{\kappa\lambda}{}^{cd} - 2i\,\varepsilon^{\mu\nu\kappa\lambda}\bar{\psi}_\mu \gamma_5 \gamma_\nu D_\mu \psi_\lambda\}}$$

$$\underline{+ \frac{7}{2} im \frac{1}{\kappa^2}\varepsilon^{\mu\nu\kappa\lambda}\bar{\psi}_\mu \gamma_5 \sigma_{\nu\kappa}\psi_\lambda}$$

$$\underline{+ 2\frac{m^2}{\kappa^2}\varepsilon^{\mu\nu\kappa\lambda}\varepsilon_{abcd} h_\mu{}^a h_\nu{}^b h_\kappa{}^c h_\lambda{}^d} \qquad (57)$$

All of these terms occurred in $\mathcal{L}_{(1)}\Big|_u$, eqn. (56). (There appears to be no clear reason for ignoring the action $\mathcal{L}_{(2)}'$ with $n = 0$, i.e. $\mathcal{L}_{(2)} = \varepsilon^{\mu\nu\kappa\lambda}\,\mathrm{Trg}\, m^2 \nabla_\mu H \nabla_\nu H F_{\kappa\lambda}$, however this action does not contain the Einstein action, rather a term $\sim \varepsilon^{\mu\nu\kappa\lambda} h_{\mu a} h_{\nu b} R_{\kappa\lambda}{}^{ab}$).

A similar calculation for $\mathcal{L}_{(3)} \to \frac{1}{\Lambda^2}\mathcal{L}_{(3)}$, gives,

$$\underline{\mathcal{L}_{(3)}\Big|_u = -\frac{3im}{\kappa^2}\varepsilon^{\mu\nu\kappa\lambda}\bar{\psi}_\mu \gamma_5 \sigma_{\nu\kappa}\psi_\lambda - 4\frac{m^2}{\kappa^2}\varepsilon^{\mu\nu\kappa\lambda}\varepsilon_{abcd} h_\mu{}^a h_\nu{}^b h_\kappa{}^c h_\lambda{}^d .}$$

(58)

It is important to remember that $\mathcal{L}_{(1)}\Big|_u$, $\mathcal{L}_{(2)}\Big|_u$ and $\mathcal{L}_{(3)}\Big|_u$ are only invariant under the local Lorentz subgroup of Osp(1,4). Choosing the unitary gauge has lost the symmetry of these actions under the gauge transformations (52). However, since we have the appropriate terms in our unitary gauge actions, we might expect to restore the local supersymmetric invariance by choosing a suitable combination of the actions. This problem is best tackled by first considering the second method for constructing spontaneously broken gauge theories of Osp(1,4), namely the method of non-linear fields. The more general method is outlined in Appendix D and applied there to the Sp(4) sub-theory. Here we extend the analysis to Osp(1,4) (Gursey and Marchildon[36]) and in doing so, make contact with the approach to supergravity of MacDowell and Mansouri[54] which enables us to identify the topological invariants.

(b) Non-Linear Lagrangians for Spontaneously Broken Gauge Theories of Osp(1,4)

Here we employ the general formalism (Appendix D) for the non-linear realization of Osp(1,4) on its 8-dimensional coset space, $\frac{Osp(1,4)}{SL(2,C)}$. (This space is isomorphic to the 8-dimensional space spanned by the constrained Higgs multiplet H, of the previous section). An element $g \in Osp(1,4)$ may be written,

$$g = ch \quad \text{where} \quad h \in SL(2,C)$$
$$c \in \frac{Osp(1,4)}{SL(2,C)} .$$

Then the group multiplication rule $(g_1 g_2 = g_3)$ implies,

$$\underline{gc = c'h_1} \quad . \quad (h_1 = h'\,h^{-1}) \quad . \tag{59a}$$

A solution to this equation would take the form $c' = c'(c,g)$ and we employ the standard exponential parameterization to obtain the explicit form of this non-linear realization of $\{g\}$ on $\{c\}$. Using this exponential parametrization we can write 'c' as,

$$\underline{c = e^{\bar{\lambda}\phi}\, e^{i y^a M_{5a}}} \qquad \text{(Zumino}^{(80)}\text{)} \tag{59b}$$

or

$$\underline{c = e^{i y^a M_{5a}}\, e^{\bar{\lambda} Q}} \;. \qquad \text{(Ivanov and Sorin}^{(45)}\text{)} \tag{59c}$$

The difference between these two ways of writing c is that in the former case (Zumino) the coset parameters, $\lambda\alpha$ form an Sp(4) Majorana spinor transforming linearly under Sp(4) as,

$$\lambda_\alpha \xrightarrow{Sp(4)} \lambda_\alpha{}' = \exp(-\tfrac{i}{4}\,\omega^{AB}\,\Sigma_{AB})_\alpha{}^\beta\,\lambda_\beta ,$$

and in the latter case (Ivanov and Sorin) λ_α is a Lorentz Majorana spinor which transforms under $\kappa \in Sp(4)$ with a non-linear element of SL(2,c),

$$\lambda_\alpha \xrightarrow{Sp(4)} \exp(-\tfrac{i}{4}\,\omega^{ab}(y_a,\kappa)\sigma_{ab})_\alpha{}^\beta\,\lambda_\beta \;.$$

(See Appendix D for more details of the non-linear realization of Sp(4) on $\frac{Sp(4)}{SL(2,c)}$) .

In either case we identify λ_α as the coordinates in the coset space, $\frac{Osp(1,4)}{Sp(4)}$ and y_a as the coordinates in the coset space $\frac{Sp(4)}{SL(2,c)}$. According to the general theory, outlined in Appendix D, λ_α and y_a are the Goldstone modes associated with the spontaneous breakdown of Osp(1,4) to SL(2,c). In the context of a local gauge theory of Osp(1,4), the fields $\lambda_\alpha(x)$ and $y_a(x)$ can be gauged away by a suitable gauge transformation, namely

$$g = c^{-1} = e^{-i y_a(x) M_{5a}}\; e^{-\bar{\lambda}(x)\phi} , \quad \text{(with Zumino's parametrization (59b).}$$

Non-Linear Fields

For any multiplet, Φ, which transforms as some realization of Osp(1,4), written as,

$$\Phi \xrightarrow{g\ \varepsilon\ \mathrm{Osp}(1,4)} \Phi' \equiv T_g\,\Phi \qquad \text{(Appendix D)}$$

then we define a 'non-linear' multiplet, $\hat{\Phi}$ by,

$$\underline{\hat{\Phi} = T_{c^{-1}}\,\Phi\ .} \qquad (60a)$$

The transformation of $\hat{\Phi}$ under $g \varepsilon \mathrm{Osp}(1,4)$ is then given by,

$$\hat{\Phi} \xrightarrow{g} \hat{\Phi}' = T_{c'^{-1}}\,\Phi' = T_{c'^{-1}}\,T_g\,\Phi\ .$$

Using (59a) and the requirement that $T_{g_1}T_{g_2} = T_{g_1 g_2}$ we obtain,

$$\hat{\Phi}' = T_{h_1 c^{-1}}\Phi$$

$$\underline{\hat{\Phi}' = T_{h_1}\,\hat{\Phi}}\ , \qquad \underline{h_1 = h_1(g,c)}\ . \qquad (60b)$$

Hence the multiplet $\hat{\Phi}$ transforms only under the subgroup SL(2,c) but with elements, $h_1 = h_1(g,c) \equiv h_1(g,\lambda_\alpha, y_a)$ which are derived from the full Osp(1,4) group element, g. In a general gauge the explicit form of equations (60) is very complicated, (see reference (36)). In the unitary gauge where $C = I$ then $\hat{\Phi} = \Phi$ and the gauge group is reduced to the stability subgroup SL(2,c) so that $h_1 = h_1(g,0,0) = g \varepsilon \mathrm{SL}(2,c)$.

Gauge Actions

The Osp(1,4) gauge potentials, $\omega_\mu = \frac{i}{2} B_\mu{}^{ab} M_{ab} + im\, h_\mu{}^a M_{5a} + \sqrt{m}\, \bar{\psi}_\mu{}^\alpha Q_\alpha$, transform under $g \in Osp(1,4)$ as,

$$\omega_\mu \xrightarrow{g} \omega_\mu{}' = Tg\, \omega_\mu = g\, \omega_\mu\, g^{-1} - g\, \partial_\mu\, g^{-1}.$$

Hence we define the non-linear potentials in $\hat{\omega}_\mu$ by

$$\underline{\hat{\omega}_\mu = T_{\mathcal{C}^{-1}}\, \omega_\mu = \mathcal{C}^{-1}\, \omega_\mu\, \mathcal{C} - \mathcal{C}^{-1}\, \partial_\mu\, \mathcal{C}} \tag{61a}$$

and we may verify directly that $\hat{\omega}_\mu$ transforms under $g \in Osp(1,4)$ as,

$$\underline{\hat{\omega}_\mu \xrightarrow{g} \hat{\omega}_\mu = T_{h_1}\, \hat{\omega}_\mu = h_1\, \hat{\omega}_\mu\, h_1{}^{-1} - h_1\, \partial_\mu\, h_1{}^{-1}} \tag{61b}$$

Now $h_1\, \partial_\mu\, h_1{}^{-1} \in$ alg SL(2,c) which, together with the decomposition, (35), of the algebra of Osp(1,4) implies that,

$$\underline{\hat{h}_\mu{}'^a M_{5a} = h_1 (\hat{h}_\mu{}^a\, M_{5a}) h_1{}^{-1} \equiv [\Lambda(h_1)^a{}_b\, \hat{h}_\mu{}^b] M_{5a}} \tag{62a}$$

and

$$\underline{\hat{\psi}_\mu{}'^\alpha Q_\alpha = h_1 (\hat{\psi}_\mu\, Q_\alpha) h_1{}^{-1} \equiv Q^\alpha\, S(h_1)_\alpha{}^\beta\, \hat{\psi}_{\mu\beta}} \tag{62b}$$

where $h \longrightarrow \Lambda(h)_a{}^b$ is the 4×4 Lorentz matrix representation of $h \in SL(2,c)$

$h \longrightarrow S(h)_\alpha{}^\beta$ is the 4×4 Dirac spinor representation of $h \in SL(2,c)$.

(see Appendix A)

We have therefore established that the 'non-linear' gauge potentials $\hat{h}_\mu{}^a$ and $\hat{\psi}_{\mu\alpha}$, defined by equation (61a) transform homogeneously under Osp(1,4) according to equations (62). It is only the non-linear SL(2,c) potentials which retain their characteristic gauge transformation properties,

$$\underline{\tfrac{i}{2}\,\hat{B}_\mu{}'^{ab}\,M_{ab} \;=\; h_1(\tfrac{i}{2}\,\hat{B}_\mu{}^{ab}\,M_{ab})h_1^{-1} - h_1\,\partial_\mu\,h_1^{-1}} \; . \qquad (62c)$$

The non-linear field strengths, $\hat{F}_{\mu\nu} = \partial_\mu\,\hat{\omega}_\nu - \partial_\nu\,\hat{\omega}_\mu - [\hat{\omega}_\mu\;\hat{\omega}\,]$, are related to the $F_{\mu\nu}$ by,

$$\underline{\hat{F}_{\mu\nu} \;=\; T_{c^{-1}}\;F_{\mu\nu} \;=\; c^{-1}\,F_{\mu\nu}\,c} \qquad (63a)$$

and hence, transform homogeneously under Osp(1,4) according to,

$$\underline{\hat{F}_{\mu\nu} \xrightarrow{g} \hat{F}_{\mu\nu} \;=\; h_1\;\hat{F}_{\mu\nu}\;h_1^{-1}} \; . \qquad (63b)$$

We may now write down Osp(1,4) invariants, formed from the non-linear gauge potentials and gauge field strengths. Because of the form (60b) of their transformation under the full group, we see that we need only form linear SL(2,c) invariants which will then be automatically invariant under the full non-linear action of Osp(1,4). In addition to the (usually) covariant field strengths, $\hat{B}_{\mu\nu}{}^{ab}$, $\hat{h}_{\mu\nu}{}^{a}$ and $\hat{\psi}_{\mu\nu\alpha}$ we may also include the covariant gauge potentials $\hat{h}_\mu{}^a$ and $\hat{\psi}_{\mu\alpha}$ explicitly into the action. It is only the SL(2,c) gauge potentials $\hat{B}_\mu{}^{ab}$ which are restricted to enter an invariant only through the field strengths. Gauge actions are constructed from Osp(1,4) invariants which are also differential 4-forms on space-time (as discussed earlier). Suitable Lagrangians therefore include,

$$\hat{\mathcal{L}}_1 \;=\; m^4\;\varepsilon^{\mu\nu\kappa\lambda}\;\varepsilon_{abcd}\;\hat{h}_\mu{}^a\,\hat{h}_\nu{}^b\,\hat{h}_\kappa{}^c\,\hat{h}_\lambda{}^d$$

$$\hat{\mathcal{L}}_2 \;=\; m^3\;\varepsilon^{\mu\nu\kappa\lambda}\;\varepsilon_{abcd}\;\hat{h}_\mu{}^a\,\hat{h}_\mu{}^b\,\hat{\bar{\psi}}_\kappa\,\sigma^{cd}\,\hat{\psi}_\lambda$$

$$\hat{\mathcal{L}}_3 = m^2 \, \varepsilon^{\mu\nu\kappa\lambda} \, \varepsilon_{abcd} \, \hat{h}_\mu{}^a \, \hat{h}_\nu{}^b \, \hat{B}_{\kappa\lambda}{}^{cd}$$

$$\hat{\mathcal{L}}_4 = m \, \varepsilon^{\mu\nu\kappa\lambda} \, \varepsilon_{abcd} \, \hat{\bar{\psi}}_\mu \, \sigma^{ab} \, \hat{\psi}_\nu \, \hat{B}_{\kappa\lambda}{}^{cd}$$

$$\hat{\mathcal{L}}_5 = \varepsilon^{\mu\nu\kappa\lambda} \, \varepsilon_{abcd} \, \hat{B}_{\mu\nu}{}^{ab} \, \hat{B}_{\kappa\lambda}{}^{cd}$$

$$\hat{\mathcal{L}}_6 = m^2 \, \varepsilon^{\mu\nu\kappa\lambda} \, \hat{h}_\mu{}^a \, \hat{\bar{\psi}}_\nu \, \gamma_5 \gamma_a \, \hat{\psi}_{\kappa\lambda}$$

$$\hat{\mathcal{L}}_7 = m \, \varepsilon^{\mu\nu\kappa\lambda} \, \hat{\bar{\psi}}_{\mu\nu} \, \gamma_5 \, \hat{\psi}_{\kappa\lambda}$$

where

$$\hat{B}_{\mu\nu}{}^{ab} = \hat{R}_{\mu\nu}{}^{db} + m^2 (\hat{h}_\mu{}^a \, \hat{h}_\nu{}^b - \hat{h}_\nu{}^a \, \hat{h}_\mu{}^b) + im \, \hat{\bar{\psi}}_\mu \, \sigma^{ab} \, \hat{\psi}_\nu$$

$$\hat{\psi}_{\mu\nu\alpha} = \hat{D}_\mu \hat{\psi}_\nu - \hat{D}_\nu \hat{\psi}_\mu - \frac{im}{2} (\hat{\gamma}_\mu \hat{\psi}_\nu - \hat{\gamma}_\nu \hat{\psi}_\mu) \quad .$$

$$(\hat{D}_\mu \equiv \partial_\mu - \frac{i}{2} \hat{B}_\mu{}^{ab} M_{ab}, \qquad \hat{\gamma}_\mu \equiv \hat{h}_\mu{}^a \gamma_a).$$

Notice that we have not included actions, such as $m^2 \, \varepsilon^{\mu\nu\kappa\lambda} \, \hat{h}_{\mu\nu}{}^a \, \hat{\bar{\psi}}_k \, \gamma_a \, \hat{\psi}_\lambda$, which involve the translational field strengths, $\hat{h}_{\mu\nu}{}^a = \hat{D}_\mu \, \hat{h}_\nu{}^a - \hat{D}_\nu \, \hat{h}_\mu{}^a - i \, \hat{\bar{\psi}}_\mu \gamma^a \, \hat{\psi}_\nu$, since we shall continue to impose $\hat{h}_{\mu\nu}{}^a = 0$ as the non-dynamical torsion constraint[9].

In the unitary gauge, $\hat{\Phi} = \Phi$, we recognise all the terms in the above actions as having occurred in our previous actions, $\mathcal{L}_{(1)}\big|_u$, $\mathcal{L}_{(2)}\big|_u$ and $\mathcal{L}_{(3)}\big|_u$. The difference is that we are now at liberty to pick any linear combination of these terms as a candidate theory for Osp(1,4) spontaneously broken to SL(2,c). In particular we could take the Einstein action (contained in $\hat{\mathcal{L}}_3$) which in a general gauge is written,

$$\hat{\mathcal{L}}_E = \frac{1}{\kappa^2} \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{abcd} \hat{h}_\mu{}^a \hat{h}_\nu{}^b \hat{R}_{\kappa\lambda}{}^{cd} .$$

Although $\hat{\mathcal{L}}_E$ doesn't contain the gravitino field $\psi_{\mu\alpha}$ it is fully invariant under Osp(1,4). Clearly we are at liberty to write down the 'non-linear supergravity Lagrangian' (from eqn. (2.46))

$$\hat{\mathcal{L}}_{SG} = \frac{1}{4\kappa^2} \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{abcd} \hat{h}_\mu{}^a \hat{h}_\nu{}^b \hat{R}_{\kappa\lambda}{}^{cd} - \frac{2i}{\kappa^2} \varepsilon^{\mu\nu\kappa\lambda} \hat{\bar{\psi}}_\mu \gamma_5 \hat{\gamma}_\nu \hat{D}_\kappa \hat{\psi}_\lambda$$

both terms of which are separately invariant under local Osp(1,4). We know however that even in the unitary gauge this Lagrangian, possesses, in addition to the required local Lorentz invariance, the local supersymmetry,

$$\delta_\varepsilon h_\mu{}^a = i \bar{\varepsilon} \gamma^a \psi_\mu \qquad \delta_\varepsilon \psi_\mu = D_\mu \varepsilon .$$

These transformations are part of the Wess-Zumino group gauge transformations (16), obtained from the Osp(1,4) gauge transformations by taking the group contraction, $m \to 0$, limit of (52). With $m \neq 0$ we see that the supergauge transformations (52) (with $\xi^a = \omega^{ab} = 0$) read,

$$\left.\begin{aligned} &\underline{\delta_\varepsilon h_\mu{}^a = i \bar{\varepsilon} \gamma^a \psi_\mu} \qquad \underline{\delta_\varepsilon \psi_\mu = D_\mu \varepsilon - \frac{im}{2} \gamma_\mu \varepsilon} \\ &\qquad\qquad \underline{\delta_\varepsilon B_\mu{}^{ab} = - im\bar{\varepsilon}\, \sigma^{ab} \psi_\mu .} \end{aligned}\right\} \qquad (64)$$

Since we require $B_\mu{}^{ab}$ to satisfy its algebraic field equation, the variation $\delta_\varepsilon B_\mu{}^{ab}$ will not affect any action variation. The variation of $\mathcal{L}_{SG}$ under (64) is then (with $\frac{\delta I_{SG}}{\delta B_\mu{}^{ab}} = 0$),

$$\underline{\delta_\varepsilon \mathcal{L}_{SG} = \text{total derivative} - \frac{1}{\kappa^2} (2im\, \varepsilon^{\mu\nu\kappa\lambda} h_\nu{}^a \bar{\psi}_\mu \gamma_5 \varepsilon \bar{\psi}_\kappa \gamma_a \psi_\lambda}$$

$$\underline{+ 2im\, \varepsilon^{\mu\nu\kappa\lambda} \bar{\psi}_\mu \gamma_5 \sigma_{\nu\kappa} D_\lambda \varepsilon).}$$

(With $m = 0$, $\delta_\varepsilon \mathcal{L}_{SG} =$ total derivative; Chapter II).

We have therefore found, not surprisingly, that the supergravity action is not invariant under the anti de Sitter supergauge transformations, (64). When $m \neq 0$ however, we know that the cosmology term $\sim \hat{\mathcal{L}}_1$ and the gravitino mass term $\sim \hat{\mathcal{L}}_2$ are non zero and we might expect that a suitable linear combination of these two terms together with $\hat{\mathcal{L}}_{SG}$ retains the supersymmetric invariance (under (64)) in the unitary gauge.

The cosmological term is, (in the unitary gauge)

$$\underline{\mathcal{L}_{cos} = \frac{1}{\Lambda^2}\mathcal{L}_1 = \frac{m^2}{\kappa^2}\varepsilon^{\mu\nu\kappa\lambda}\varepsilon_{abcd}\, h_\mu{}^a h_\nu{}^b h_\kappa{}^c h_\lambda{}^d}$$

$$\therefore \quad \delta_\varepsilon \mathcal{L}_{cos} = \frac{4m^2}{\kappa^2}\varepsilon^{\mu\nu\kappa\lambda}\varepsilon_{abcd}(i\,\bar{\varepsilon}\,\gamma^a\psi_\mu)h_\nu{}^b h_\kappa{}^c h_\lambda{}^d$$

but $\varepsilon^{\mu\nu\kappa\lambda}\varepsilon_{abcd}\, h_\nu{}^b h_\kappa{}^c h_\lambda{}^d = 6h\, h^\mu{}_a$ (Appendix C)

$$\therefore \quad \underline{\delta_\varepsilon \mathcal{L}_{cos} = \frac{24im^2}{\kappa^2}\, h\, h^\mu{}_a\, \bar{\varepsilon}\,\gamma^a\psi_\mu} \; .$$

The gravitino mass term is,

$$\underline{\mathcal{L}_{mass} = \frac{i}{2\Lambda^2}\mathcal{L}_2 = \frac{im}{\kappa^2}\varepsilon^{\mu\nu\kappa\lambda}\,\bar{\psi}_\mu\,\gamma_5\,\sigma_{\nu\kappa}\,\psi_\lambda}$$

and $\delta_\varepsilon \mathcal{L}_{mass}$, calculated from (64) is found to be

$$\underline{\delta_\varepsilon\mathcal{L}_{mass} = \frac{2im}{\kappa^2}\varepsilon^{\mu\nu\kappa\lambda}\,\bar{\psi}_\mu\gamma_5\,\sigma_{\nu\kappa}\,D_\lambda\varepsilon + \frac{6im^2}{\kappa^2}\,\bar{\psi}_\mu\,\gamma^a\,\varepsilon\, h\, h^\mu{}_a}$$

$$\underline{- \frac{2m}{\kappa^2}\varepsilon^{\mu\nu\kappa\lambda}\,\bar{\psi}_\mu\,\gamma_5\,\sigma_{ab}\,\psi_\lambda\, h_\kappa{}^b\,\bar{\varepsilon}\,\gamma^a\,\psi_\nu} \; .$$

We form a supergravity Lagrangian with cosmological and gravitino mass terms from the linear combination,

$$\underline{\mathcal{L} = \mathcal{L}_{SG} + \alpha_1 \mathcal{L}_{cos} + \alpha_2 \mathcal{L}_{mass}}$$

where α_1 and α_2 are dimensionless constants, to be fixed by the requirement that $\int \mathcal{L}\, d^4x$ is invariant under (64). From our preceding results we see that

$$\begin{aligned}\delta_\varepsilon \mathcal{L} = & \text{ total derivative } - \frac{2im}{\kappa^2} \varepsilon^{\mu\nu\kappa\lambda} \bar{\psi}_\mu \gamma_5 \sigma_{\nu\kappa} D_\lambda \varepsilon (1 - \alpha_2) \\ & + \frac{24im^2}{\kappa^2} h h^\mu{}_a \bar{\varepsilon} \gamma^a \psi_\mu (\alpha_1 - \tfrac{1}{4}\alpha_2) \\ & - \frac{2im}{\kappa^2} \varepsilon^{\mu\nu\kappa\lambda} \bar{\psi}_\mu \gamma_5 \varepsilon \bar{\psi}_\kappa \gamma_a \psi_\lambda h_\nu{}^a \\ & - \frac{2m}{\kappa^2} \alpha_2 \varepsilon^{\mu\nu\kappa\lambda} \bar{\psi}_\mu \gamma_5 \sigma_{ab} \psi_\lambda h_\kappa{}^b \bar{\varepsilon} \gamma^a \psi_\nu .\end{aligned}$$

The first two terms vanish if, $\underline{\alpha_2 = 1}$ and $\underline{\alpha_1 = \frac{1}{4}\alpha_2 = \frac{1}{4}}$

After two Fierz resummations we find that,

$$\varepsilon^{\mu\nu\kappa\lambda} \bar{\psi}_\mu \gamma_5 \sigma_{ab} \psi_\lambda \bar{\psi}_\nu \gamma^a \varepsilon h_\kappa{}^b = i \varepsilon^{\mu\nu\kappa\lambda} \bar{\psi}_\mu \gamma_5 \varepsilon \bar{\psi}_\lambda \gamma_b \psi_\nu h_\kappa{}^b$$

hence, with $\alpha_2 = 1$ the remaining two terms cancel.
Thus $\delta_\varepsilon \mathcal{L}$ = total derivative, when

$$\begin{aligned}\mathcal{L} = & \frac{1}{4\kappa^2} \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{abcd} h_\mu{}^a h_\nu{}^b R_{\kappa\lambda}{}^{cd} - \frac{2i}{\kappa^2} \varepsilon^{\mu\nu\kappa\lambda} \bar{\psi}_\mu \gamma_5 \gamma_\nu D_\kappa \psi_\lambda \\ & + \frac{im}{\kappa^2} \varepsilon^{\mu\nu\kappa\lambda} \bar{\psi}_\mu \gamma_5 \sigma_{\nu\kappa} \psi_\lambda + \frac{6m^2}{\kappa^2} \det h_{\mu a} .\end{aligned} \tag{65}$$

This Lagrangian $\mathcal{L}$ is a particular combination of terms, each of which is invariant under the full Osp(1,4) group (realized non-linearly) in a general gauge, which retains its anti de Sitter supersymmetry even in the unitary gauge. The field equation for the gravitino, ψ_μ, calculated from (65) is, (see also Appendix C)

$$\frac{\delta I}{\delta\bar{\psi}^{\mu}} = \varepsilon^{\mu\nu\kappa\lambda}\gamma_5\gamma_\nu(D_\kappa - \frac{im}{2}\gamma_\kappa)\psi_\lambda = 0 \ . \tag{66}$$

Deser and Zumino[17] studied this wave equation in space-time manifolds with a cosmological constant and found that it gave the correct description for a massless spin $\frac{3}{2}$ particle, provided the cosmological constant is $\Lambda = 6m^2$. This is the case for our Lagrangian (65) so we see that supergravity theory in space-time with a cosmological constant is still a theory with a massless gravitino. (The mass term, $\sim m = \frac{1}{R}$ was further explored by Zumino in a separate publication[80] where he showed that massless, spin $\frac{1}{2}$ Fermions in anti de Sitter space-time would require kinetic terms, $i\bar{\psi}\partial\psi - m\bar{\psi}\psi$. Clearly as $m \to 0$ (group contraction) the usual kinetic term of the Poincaré invariant field theories is recovered). If we were to choose different constants, α_1 and α_2 in (65), for instance $\alpha_1 = 0$ so that there would be no cosmology term, then the ψ_μ would be a massive gravitino field but we should have lost the supersymmetric invariance under (64). Although $\mathcal{L}$, given by (65), is invariant under local Lorentz and local supersymmetry transformations, it is not invariant under the $\frac{Sp(4)}{SL(2,c)}$ coset translations, parametrized by ξ^a, in the Osp(1,4) gauge transformations (52). This follows directly from our result, equation (20), in the first section which shows that even the contracted theory fails to have invariance under ξ^a translations. (Clearly $O(m)$ corrections cannot remove the terms in (20) so that we need not calculate $\delta_\xi\mathcal{L}$ to know that it will <u>not</u> be a total derivative).

Topological Invariants

We return now to the problem of identifying topological invariants constructed from the Osp(1,4) gauge fields. This must be done so that we may take care of the unwanted '$\frac{1}{m}$ terms' in the Lagrangians $\mathcal{L}_{(1)}\big|_u$ and $\mathcal{L}_{(2)}\big|_u$ (equations (56) and (57)). We continue to work in the unitary gauge with actions which are only invariant under the Lorentz subgroup. An approach to local supersymmetry based on such Lorentz invariant actions was considered by MacDowell and Mansouri[(54)]. They constructed an action from the Osp(1,4) field strengths, (53), which was required to be a topological invariant when the theory was restricted to Osp(1,2c). As we mentioned earlier, this restriction is achieved simply by removing the generators, M_{5a} and their associated gauge potentials $mh_\mu{}^a$ and field strengths $m\,h_{\mu\nu}{}^a$ from any Osp(1,4) relation. In particular we see from (53) that the Osp(1,2c) field strengths are

$$\underline{{}^*B_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} + im\,\bar{\psi}_\mu\,\sigma^{ab}\,\psi_\nu} \qquad (67a)$$

and

$$\underline{{}^*\psi_{\mu\nu} = D_\mu\,\psi_\nu - D_\nu\,\psi_\mu} \qquad (67b)$$

The class of actions considered by MacDowell and Mansouri was restricted to

$$\underline{\mathcal{L} = \varepsilon^{\mu\nu\kappa\lambda}\,F_{\mu\nu}{}^i\,F_{\kappa\lambda}{}^j\,N_{ij}} \qquad (68)$$

where: $F_{\mu\nu}{}^i = \{i\,B_{\mu\nu}{}^{ab},\ imh_{\mu\nu}{}^a,\ \sqrt{m}\,\bar{\psi}^\alpha_{\mu\nu}\}$ are the Osp(1,4) field strengths and $N_{ij} = -(-1)^{\sigma_i\sigma_j}\,N_{ji}$ are constants. (See the first section of this chapter for the notation employed here).

These gauge actions, (68) are required only to have local Lorentz invariance. Under a general element,

$$\varepsilon = \varepsilon^i X_i = \frac{i}{2}\,\omega^{ab}\,M_{ab} + im\,\xi^a\,M_{5a} + \sqrt{m}\,\bar{\varepsilon}^\alpha Q_\alpha$$

of the algebra of Osp(1,4) we see using (10) that,

$$\underline{\delta_\varepsilon \mathcal{L} = 2\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon^k\,f_{k\ell}{}^i\,F_{\mu\nu}{}^\ell\,F_{\kappa\lambda}{}^j\,N_{ij}} \tag{69}$$

To identify topological invariants, we require the variation of the action (68) under general variations of the gauge fields,

$$\omega_\mu i = \{i\,B_\mu{}^{ab},\ imh_\mu{}^a,\ \sqrt{m}\,\bar{\phi}_\mu{}^\alpha\}\ . \qquad \text{Now, since}$$

$$F_{\mu\nu}{}^i = \partial_\mu\omega_\nu{}^i - \partial_\nu\omega_\mu{}^i - f_{jk}{}^i\,\omega_\mu{}^j\,\omega_\nu{}^k$$

then $\delta_\omega F_{\mu\nu}{}^i = \partial_\mu(\delta\,\omega_\nu{}^i) - \partial_\nu(\delta\omega_\mu{}^i) - 2f_{jk}{}^i(\delta\,\omega_\mu{}^j)\omega_\nu{}^k$

and $\varepsilon^{\mu\nu\kappa\lambda}\,\delta_\omega F_{\mu\nu}{}^i = 2\,\varepsilon^{\mu\nu\kappa\lambda}\left[\partial_\mu(\delta\omega_\nu{}^i) - f_{jk}{}^i(\delta\omega_\mu{}^j)\omega_\nu{}^k\right]$

Substituting this into (68) we see that,

$$\delta_\omega \mathcal{L} = 4\,\varepsilon^{\mu\nu\kappa\lambda}\left[\partial_\mu(\delta\omega_\nu{}^i) - f_{mn}{}^i(\delta\omega_\mu{}^m)\mu_\nu{}^n\right]F_{\kappa\lambda}{}^j N_{ij}\ .$$

Integrating the first term by parts and using the Jacobi identity (11),

$\varepsilon^{\mu\nu\kappa\lambda}\,\nabla_\mu F_{\kappa\lambda}{}^j \equiv 0$, which implies $\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu F_{\kappa\lambda}{}^j = \varepsilon^{\mu\nu\kappa\lambda} f_{mn}{}^j\,\omega_\mu{}^m\,F_{\kappa\lambda}{}^n$,

we find that,

$$\delta_\omega \mathcal{L} = \text{total derivative} - 4\varepsilon^{\mu\nu\kappa\lambda}\left[(\delta\omega_\nu{}^i)f_{mn}{}^j\omega_\mu{}^m F_{\kappa\lambda}{}^n + f_{mn}{}^i\omega_\mu{}^m(\delta\omega_\nu{}^n)F_{\kappa\lambda}{}^j\right]N_{ij}.$$

Rearranging terms we see that this may be written,

$$\delta_\omega \mathcal{L} = \text{total derivative} + 4\varepsilon^{\mu\nu\kappa\lambda}(\delta\omega_\mu{}^i)\omega_\nu{}^j F_{\kappa\lambda}{}^m(f_{jm}{}^n N_{in} - f_{ij}{}^n\,N_{nm})\,. \tag{70}$$

The condition that $\mathcal{L}$ is the integrand of a topological invariant is then

$$f_{jm}{}^n\,N_{in} - f_{ij}{}^n\,N_{nm} = 0\ . \tag{71}$$

A simple example is provided by the action for the Lorentz group where $\omega_\mu{}^i = \{i\ B_\mu{}^{ab}\}$. The action (68) becomes the Gauss-Bonnet topological invariant, $-\frac{1}{4}\varepsilon^{\mu\nu\kappa\lambda} R_{\mu\nu}{}^{ab} R_{\kappa\lambda}{}^{cd}\varepsilon_{abcd}$ when we choose, $N_{ij} \equiv N_{ab\ cd} = \varepsilon_{abcd}$ and we may verify directly that,

$$f_{cd}{}^{gh}{}_{ef}\ \varepsilon_{abgh} - f_{ab}{}^{gh}{}_{cd}\ \varepsilon_{ghef} = 0$$

by substitution of the SL(2,c) structure constants from equation (13a). For the group Osp(1,2c), MacDowell and Mansouri identified the topological invariant,

$$^{*}\mathcal{L} = \varepsilon^{\mu\nu\kappa\lambda}\left[-\frac{1}{4}\,{}^{*}B_{\mu\nu}{}^{ab}\,{}^{*}B_{\kappa\lambda}{}^{cd}\,\varepsilon_{abcd} + am\,{}^{*}\bar{\psi}_{\mu\nu}{}^{\alpha}\,{}^{*}\bar{\psi}_{\kappa\lambda}{}^{\beta}(\gamma_5 c)_{\alpha\beta}\right] \tag{72}$$

where 'a' is a dimensionless constant which will be fixed by the requirement that (71) holds. Comparison with (68) shows that $^{*}\mathcal{L}$ is formed with $\{N_{ij};\ N_{ab\ cd} = \varepsilon_{abcd},\ N_{\alpha\beta} = a(\gamma_5 c)_{\alpha\beta},\ N_{ab\alpha} = 0\}$.

The non vanishing structure constants of Osp(1,2c) are obtained from (35) with $mM_{5a} \equiv P_a = 0$,

$$f_{ab}{}^{ef}{}_{cd} = \text{SL(2,c) structure constants, (13a)},$$

$$f_{ab\alpha}{}^{\beta} = \tfrac{1}{2}(\sigma_{ab})_\alpha{}^\beta,$$

$$f_{\alpha\beta}{}^{ab} = (\sigma^{ab}c)_{\alpha\beta}$$

all other $f_{ij}{}^k = 0$.

We therefore see that, for $^{*}\mathcal{L}$ the conditions (71) become

$$\text{(i)}\quad f_{cd}{}^{gh}{}_{ef}\ \varepsilon_{abgh} - f_{abcd}{}^{gh}\ \varepsilon_{ghef} = 0$$

and

$$\text{(ii)}\quad f_{cd\alpha}{}^{\gamma}\ a(\gamma_5 c)_{\beta\gamma} - f_{\beta cd}{}^{\gamma}\ a(\gamma_5 c)_{\gamma\alpha} = 0$$

and

$$\text{(iii)}\quad \tfrac{1}{2}f_{\alpha\beta}{}^{ab}\ \varepsilon_{cdab} - f_{cd\alpha}{}^{\gamma}\ a(\gamma_5 c)_{\gamma\beta} = 0\ .$$

Condition (i) has just been discussed in connection with the Gauss-Bonnet topological invariant and we know that it holds. Condition (ii) reads,

$$\frac{a}{2}(\sigma_{cd})_{\alpha}{}^{\gamma}(\gamma_5 c)_{\beta\gamma} + \frac{a}{2}(\sigma_{cd})_{\beta}{}^{\gamma}(\gamma_5 c)_{\gamma\alpha} = 0$$

$$\text{i.e.} \quad \frac{a}{2}(-(\sigma_{cd}\gamma_5 c)_{\alpha\beta} + (\sigma_{cd}\gamma_5 c)_{\beta\alpha}) = 0.$$

But $\quad (\sigma_{cd}\gamma_5 c)^T = \sigma_{cd}\gamma_5 c \quad ,$

hence (ii) holds identically.

Finally, condition (iii) reads,

$$-\frac{1}{2}(\sigma^{ab}c)_{\alpha\beta}\,\varepsilon_{abcd} - \frac{a}{2}(\sigma_{cd})_{\alpha}{}^{\gamma}(\gamma_5 c)_{\gamma\beta} = 0$$

but $\varepsilon_{abcd}\,\sigma^{ab} = 2\,\gamma_5\,\sigma_{cd}$ so that the left hand side becomes,

$$-(\sigma_{cd}\,\gamma_5 c)_{\alpha\beta} - \frac{a}{2}(\sigma_{cd}\,\gamma_5\,c)_{\alpha\beta} \quad .$$

Hence condition (iii) is satisfied iff $\underline{a = -2}$, in which case ${}^*\mathcal{L}$ becomes,

$$\underline{{}^*\mathcal{L} = -\frac{1}{4}\,\varepsilon^{\mu\nu\kappa\lambda}\varepsilon_{abcd}\,{}^*B_{\mu\nu}{}^{ab}\,{}^*B_{\kappa\lambda}{}^{cd} + 2m\,\varepsilon^{\mu\nu\kappa\lambda}\,{}^*\bar{\psi}_{\mu\nu}\,\gamma_5\,{}^*\psi_{\kappa\lambda}.} \qquad (73a)$$

Substituting for the field strengths from (67) and using the identity (55) to drop the quartic terms in ψ_μ we obtain,

$$\underline{{}^*\mathcal{L} = -\frac{1}{4}\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,R_{\mu\nu}{}^{ab}\,R_{\kappa\lambda}{}^{cd} - \frac{im}{2}\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,\bar{\psi}_\mu\,\sigma^{ab}\,\psi_\nu\,R_{\kappa\lambda}{}^{cd}}$$

$$\underline{+\,8m\,\varepsilon^{\mu\nu\kappa\lambda}\,\overline{D_\mu\psi_\lambda}\,\gamma_5\,D_\kappa\psi_\lambda} \qquad (73b)$$

If the field strengths in (73a) become the full Osp(1,4) quantities (53) then ${}^*\mathcal{L}$ is no longer a topological invariant. In fact we see that by 'dropping the stars' in (73a) we obtain,

$$*\mathcal{L} \rightarrow \mathcal{L} = *\mathcal{L} - \varepsilon^{\mu\nu\kappa\lambda}\, \varepsilon_{abcd} \left[m^2 h_\mu{}^a\, h_\nu{}^b\, R_{\kappa\lambda}{}^{cd} + m^4\, h_\mu{}^a\, h_\nu{}^b\, h_\kappa{}^c\, h_\lambda{}^d \right.$$

$$\left. + \; im^3\, h_\mu{}^a\, h_\nu{}^b\, \bar{\psi}_\kappa\, \sigma^{cd}\, \psi_\lambda \right]$$

$$+ \; 2m\, \varepsilon^{\mu\nu\kappa\lambda} \left[4im\, \bar{\psi}_\mu\, \gamma_5\, \gamma_\nu\, D_\kappa\, \psi_\lambda + m^2\, \bar{\psi}_\mu\, \gamma_5\, \gamma_\nu\, \gamma_\kappa\, \psi_\lambda \right].$$

Dropping the $*\mathcal{L}$ and multiplying by $\frac{1}{\Lambda^2}$ (with $\kappa = \frac{\Lambda}{m}$) we find,

$$\underline{\mathcal{L} = -\frac{1}{\kappa^2}\, \varepsilon^{\mu\nu\kappa\lambda}\, \varepsilon_{abcd}\, h_\mu{}^a\, h_\nu{}^b\, R_{\kappa\lambda}{}^{cd} + \frac{8i}{\kappa^2}\, \varepsilon^{\mu\nu\kappa\lambda}\, \bar{\psi}_\mu\, \gamma_5\, \gamma_\nu\, D_\kappa\, \psi_\lambda}$$

$$\underline{- \frac{4im}{\kappa^2}\, \varepsilon^{\mu\nu\kappa\lambda}\, \bar{\psi}_\mu\, \gamma_5\, \sigma_{\nu\kappa}\, \psi_\lambda - \frac{24m^2}{\kappa^2}\, \det h_{\mu a}} \quad . \tag{74}$$

This is the Lagrangian, (65) which we obtained by requiring invariance under the supersymmetry transformations (64). This invariance is readily checked by evaluating (69) for $\varepsilon^k \rightarrow \bar{\varepsilon}^\alpha$.

We find that $\delta_{\bar{\varepsilon}^\alpha} \mathcal{L} = 0$ provided $F_{\mu\nu}{}^a = 0$ and this condition is the torsion constraint.

Of particular interest with regards to our earlier work is the topological invariant $*\mathcal{L}$. Since the first term in $*\mathcal{L}$ is the Gauss-Bonnet topological invariant it follows that the second two terms also form a topological invariant. These two terms are not separately topological invariants however since we required a particular ratio, the constant 'a' in (72), between them. If we return now to the Lagrangians $\mathcal{L}_{(1)}\big|_u$ and $\mathcal{L}_{(2)}\big|_u$ (equations (56) and (57)) obtained through the constrained Higgs field approach, we see that both terms in this topological invariant occur in $\mathcal{L}_{(1)}\big|_u$, though not in the required combination, and that just one of the terms occurs in $\mathcal{L}_{(2)}\big]_u$, Hence, in order to remove the '$\frac{1}{m}$ terms' from $\mathcal{L}_{(1)}\big]_u$ and $\mathcal{L}_{(2)}\big]_u$ we must choose the appropriate linear combination of these two actions. By inspection of (56) and (57) we see that the correct choice is,

$$\underline{\mathcal{L}_{(1)}\big]_u + 2\,\mathcal{L}_{(2)}\big]_u = \frac{1}{m\kappa^2}\{\,\varepsilon^{\mu\nu\kappa\lambda}\,\overline{D_\mu\,\psi_\nu}\,\gamma_5\,D_\kappa\,\psi_\lambda - \frac{i}{4}\,\varepsilon^{\mu\nu\kappa\lambda}\varepsilon_{abcd}\,\bar{\psi}_\mu\sigma^{ab}\psi_\nu R_{\kappa\lambda}{}^{cd}}$$

$$\underline{+\frac{1}{\kappa^2}\,\varepsilon^{\mu\nu\kappa\lambda}\varepsilon_{abcd}\,h_\mu{}^a\,h_\nu{}^b\,R_{\kappa\lambda}{}^{cd}}$$

$$\underline{+\frac{4im}{\kappa^2}\,\varepsilon^{\mu\nu\kappa\lambda}\,\bar{\psi}_\mu\,\gamma_5\,\sigma_{\nu\kappa}\,\psi_\lambda}$$

$$\underline{+\frac{3m^2}{\kappa^2}\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,h_\mu{}^a\,h_\nu{}^b\,h_\kappa{}^c\,h_\lambda{}^d}\;. \tag{75}$$

The $\frac{1}{m}$ terms may now be identified with the second two terms, in ${}^*\mathcal{L}$ and hence dropped as topological invariants. We have found, however, that our action no longer contains a kinetic term for the gravitino field, ψ_μ. This appears to be a fairly serious problem with the constrained Higgs field method of symmetry breaking since the only remaining freedom which we have with the unitary gauge action is to add a multiple of $\mathcal{L}_{(3)}\big]_\mu$, given by (58), which clearly doesn't contain the Rarita-Schwinger action for ψ_μ.

5) <u>Conclusions</u>

The problem of identifying supergravity as a gauge theory appears to be no more or less difficult than the problem for gravity alone. The invariance of the supergravity Lagrangian of Chapter II may be viewed as the gauge invariance of the graded Poincaré (Wess-Zumino) group. We reviewed this in the first section and pointed out that the supergravity

Lagrangian is not invariant under the translational part of the Poincaré group. In Chapter I we showed that the required geometrical objects for the Einstein-Cartan theory of gravity emerged in a spontaneously broken gauge theory of SO(2,3). The principal aim of the work carried out in this chapter was to examine the minimal supergravity theory based on the spontaneously broken gauge theory of SO(2,3). The most general gauge actions for this spontaneously broken gauge theory of Osp(1,4) are the non-linear gauge actions where the full Osp(1,4) invariance is achieved with the inclusion (through eqn. (61a)) of the eight Goldstone fields, $y_a(x)$ and $\lambda_\alpha(x)$. Gauging away these fields leaves only a Lorentz invariant action. However we did find that the anti de Sitter supersymmetry could be recovered in this gauge for a suitable linear combination, (65) of these Lorentz invariant terms. These supersymmetry transformations (64), although part of the Osp(1,4) gauge transformations, (52) don't transform away from the unitary gauge. We therefore see that the $\frac{Osp(1,4)}{SL(2,c)}$ sector of the spontaneously broken gauge theory, the local group elements of which are parametrized by the $y_a(x)$ and $\lambda_\alpha(x)$, is distinct from the spinorial transformations of 'supergravity theory' (Poincaré or anti de Sitter) which retain their inhomogeneous character, (i.e. with $\partial_\mu \varepsilon_\alpha$ terms). The translational invariance, under the generators M_{5a}, can only be realized non-linearly through the introduction of the Goldstone field $y_a(x)$. Unlike the spinorial transformations, this invariance is completely lost after group contraction to the Wess-Zumino group. (This is a consequence of the non-linear realization of SO(2,3) on $\frac{SO(2,3)}{SO(1,3)}$ becoming linear for ISO(1,3) on $\frac{ISO(1,3)}{SO(1,3)}$, which loses the geometrical interpretation of the vierbein fields - see Chapter I).

One clear point which has emerged in this chapter is the value of our rescaling of the fields in the adjoint representation of Osp(1,4) by the group contraction parameter, $m = \frac{1}{R}$ according to (51a). This has enabled us to identify the mass term for the gravitino as proportional to m and the cosmological term as proportional to m^2 which allowed, via the Lagrangian (65), a direct comparison of the gauge theory approach to the earlier work of Deser and Zumino[17] on massive (broken) supergravity.

We examined the constrained Higgs field approach[7],[8] to the spontaneously broken gauge theory of Osp(1,4) and found that in order to identify terms which were factored by $\frac{1}{m}$ as topological invariants, the choice of our gauge action was limited to a set, $\mathcal{L}_{(1)} + 2\mathcal{L}_{(2)} + a\mathcal{L}_{(3)}$, which doesn't contain the gravitino kinetic term, $\varepsilon^{\mu\nu\kappa\lambda}\,\bar{\psi}_\mu\gamma_5\gamma_\nu D_\kappa\psi_\lambda$. Clearly this result relies on the fact that the actions $\mathcal{L}_{(1)}$ and $\mathcal{L}_{(2)}$ of equations (54) are the only ones which may contain this kinetic term, yet this does seem to be the case.

This difficulty apart, there appears to be no particular advantage in the constrained Higgs field approach over the more general approach of constructing non-linear actions in so far as determining gauge actions is concerned. However when we consider the coupling of matter fields to supergravity we are no longer working with just the gauge multiplet and the explicit construction of Osp(1,4) actions as the graded trace of various covariant quantities is a comparatively systematic approach. It is this explicit construction of possible 'matter actions' which is the subject of the next chapter.

CHAPTER IV

MATTER COUPLING TO Osp(1,4) SUPERGRAVITY

Minimal, Osp(1,4) supergravity theory couples the spin 2 graviton to the spin $\frac{3}{2}$ gravitino and involves no other particle states. The introduction of lower spin states into the theory, without losing the symmetries of supergravity theory may be done in one of two ways:

(a) Extend the gauge group Osp(1,4) to Osp(N,4), where $N \leq 8$ (Chapter II), to obtain an extended supergravity theory. For example, Osp(2,4) is a theory with one spin 2 graviton, two spin $\frac{3}{2}$ gravitinos and one spin 1 photon with an O(2) internal gauge symmetry.

(b) Introduce matter multiplets as irreducible representations of Osp(1,4), which remains as the gauge group, and couple the matter fields to minimal supergravity via the formation of Osp(1,4) covariant derivatives.

Extended supergravity theories have many promising features, not least that each one has a uniquely determined particle content (in the adjoint representation) with gauged internal symmetries, and have received much attention in the past five years, (see van Nieuwenhuizen[58] for a review and an extensive list of references). We shall remain with the group Osp(1,4) and explore the possible actions which may be obtained through the second approach (b). In constructing actions for matter coupled to supergravity we must choose various representations of Osp(1,4) and form invariant actions from these covariants. Given that supergravity is a spontaneously broken gauge theory of Osp(1,4), as discussed in the previous chapter, we must construct Osp(1,4) invariant actions with the appropriate structure. Our interest is largely in the approach using the constrained Higgs multiplet, H of equation (3.42). However, non-linear actions for matter coupling can be written down once we have established the transformations $\Phi \to T(g)\Phi$ $(g \in Osp(1,4))$, of the matter fields Φ. We shall briefly examine the impact that the non-linear field approach has on the problem of coupling matter to supergravity.

1) Matter Multiplets, Invariant Actions and Their Flat Space-Time Reduction

The smallest representation of Osp(1,4) is the 5-dimensional fundamental vector representation,

$$V = \begin{pmatrix} \phi \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix} \qquad \begin{array}{ll} \phi & \text{is a Bose variable} \\ \chi_\alpha & \text{are Fermi variables,} \\ & \{\chi_\alpha, \chi_\beta\} = 0. \end{array}$$

The infinitesimal transformation of V under an element A of the algebra of Osp(1,4) is simply,

$$\delta_A V = A V$$

where
$$A = \left(\begin{array}{c|c} 0 & -\sqrt{m}\,\bar{\varepsilon} \\ \hline \sqrt{m}\,\varepsilon & -\frac{i}{4}\omega^{ab}\sigma_{ab} + \frac{im}{2}\xi^a\gamma_a \end{array}\right) . \qquad \text{(eqn. (3.49))}$$

In terms of the components ϕ and χ_α this transformation reads,

$$\underline{\delta_A \phi = -\sqrt{m}\,\varepsilon\,\chi} \tag{1a}$$

$$\underline{\delta_A \chi = \sqrt{m}\,\phi\,\varepsilon - \frac{i}{4}\,\omega^{ab}\,\sigma_{ab}\,\chi + \frac{im}{2}\,\xi^a\,\gamma_a\,\chi} \tag{1b}$$

As was pointed out in Chapter III (page 153) this vector representation, consisting of an SO(2,3) scalar ϕ and spinor χ_α, is difficult to accommodate in a field theory. The problem is that the nature of the transformations (1) require the fields $\phi(x)$ and $\chi_\alpha(x)$ to have the same dimension. We must therefore rescale one of the fields, for example,

$$\chi \to b\chi \qquad \text{where} \qquad [b] = (\text{mass})^{-\frac{1}{2}}$$

then with $[V] = [\phi] = $ mass, we see that $[\chi] = (\text{mass})^{3/2}$ as required. It is the introduction of this dimensionful constant, 'b' which is so problematical since there appears to be no plausible quantity available.

This is not the case for the 14-dimensional adjoint multiplet, M of Osp(1,4), where the group contraction parameter $m = \frac{1}{R}$ is introduced in the manner described in the previous chapter (cf. eqn. (3.51)),

$$\underline{M = \left(\begin{array}{c|c} 0 & -\sqrt{m}\,\overline{\chi} \\ \hline \sqrt{m}\,\chi & -\frac{i}{4}F^{ab}\sigma_{ab} + \frac{im}{2}A^a\gamma_a \end{array}\right)} \tag{2}$$

The infinitesimal transformation of M under $A \in$ Osp(1,4) is

$$\delta_A M = [A, M] \quad .$$

In component form this equation becomes,

$$\underline{\delta_A A^a = \omega^a{}_b A^b - \xi^b F^a{}_b + i\,\overline{\varepsilon}\,\gamma^a \chi} \tag{3a}$$

$$\underline{\delta_A F^{ab} = \omega^a{}_c F^{cb} + \omega^b{}_c F^{ac} - m^2(\xi^a A^b - \xi^b A^a) - im\,\overline{\varepsilon}\,\sigma^{ab}\chi} \tag{3b}$$

$$\underline{\delta_A \chi = -\frac{i}{4}\omega^{ab}\sigma_{ab}\chi + \frac{im}{2}\xi^a\gamma_a\chi + \frac{i}{4}F^{ab}\sigma_{ab}\varepsilon - \frac{im}{2}A^a\gamma_a\varepsilon} \tag{3c}$$

We see that the multiplet M contains an $SO(1,3)$ vector, A^a with the canonical dimension $[A^a]$ = mass provided $[M]$ = $(\text{mass})^2$. M also contains the spinor χ_α again with the canonical dimension $[\chi]$ = $(\text{mass})^{3/2}$ provided $[M]$ = $(\text{mass})^2$. The antisymmetric $SO(1,3)$ tensor F^{ab} has the dimensions $(\text{mass})^2$, however we would not expect a rank 2 antisymmetric tensor to describe a fundamental particle field and anticipate its role as an auxiliary field.

The adjoint multiplet M is the only representation into which we may introduce the contraction parameter m as something other than a completely arbitrary rescaling parameter. This implies that we shall have problems with the dimensions of the field in multiplets other than the vector multiplet V. We shall therefore limit our analysis to the adjoint and vector multiplets since they appear to provide a crucial test of the approach.

Matter Actions

We begin with the adjoint multiplet, M and construct an action which includes kinetic terms for the fields in M. These kinetic terms must appear within the covariant,

$$\nabla_\mu M \nabla_\nu M$$

where

$$\nabla_\mu M = \partial_\mu M - [\omega_\mu, M] \equiv \left(\begin{array}{c|c} 0 & -\sqrt{m}\,\overline{\nabla_\mu \chi} \\ \hline \sqrt{m}\,\nabla_\mu \chi & -\frac{i}{4}\nabla_\mu F^{ab}\sigma_{ab} + \frac{im}{2}\nabla_\mu A^a \gamma_a \end{array}\right) \tag{4}$$

so that, (using (3.51a))

$$\nabla_\mu A_a = D_\mu A_a + F_{ab} h_\mu{}^b - i\,\bar\psi_\mu \gamma_a \chi \tag{5a}$$

$$\nabla_\mu F^{ab} = D_\mu F^{ab} + m^2(h_\mu{}^a A^b - h_\mu{}^b A^a) + im\,\bar\psi_\mu \sigma^{ab} \chi \tag{5b}$$

$$\nabla_\mu \chi = D_\mu \chi - \frac{im}{2}\gamma_\mu \chi - \frac{i}{4} F^{ab}\sigma_{ab}\psi_\mu + \frac{im}{2} A^a \gamma_a \psi_\mu \tag{5c}$$

The Lagrangian density must therefore be a differential 4-form of the type,

$$\mathcal{L}_M \sim \varepsilon^{\mu\nu\kappa\lambda}\, \mathrm{Trg}(mH)^n\, \nabla_\mu(mH)\nabla_\nu(mH)\nabla_\kappa M\, \nabla_\lambda M$$

where H is the Higgs multiplet (3.42),

$$H = \left(\begin{array}{c|c} 4\phi & \bar\lambda \\ \hline \lambda & \phi + i\Gamma^A y_A \end{array}\right).$$

subject to the constraints (3.43),

$$\mathrm{Trg}\, H^2 = 4R^2 \qquad \text{and} \qquad \mathrm{Trg}\, H^3 = 0\,.$$

We shall work in the unitary gauge where, (equations (3.47))

$$mH\Big|_u = -\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \gamma_5 \end{array}\right) \qquad \text{and} \qquad \nabla_\mu(mH)\Big|_u = \left(\begin{array}{c|c} 0 & -\bar\psi_\mu \gamma_5 \\ \hline -\gamma_5\psi_\mu & i\Gamma^a B_{\mu a} \end{array}\right)$$

and we need only consider $n = 0, 1$ in $\mathcal{L}_M$, as was the case for the gauge actions of Chapter III. Explicit calculation shows that $n = 0$ doesn't produce any recognisable terms and that the relevant action for the adjoint multiplet is of the same form as that of Chamseddine[8],

$$\underline{\mathcal{L}_M = Z\, m^3\, \varepsilon^{\mu\nu\kappa\lambda}\ \mathrm{Trg}\ H\,\nabla_\mu\, H\,\nabla_\nu\, H\,\nabla_\kappa\, M\,\nabla_\lambda\, M} \qquad (6)$$

where Z is a constant with $[Z] = (\mathrm{mass})^{-4}$ so that $[\mathcal{L}_M] = (\mathrm{mass})^4$. In the unitary gauge,

$$\mathcal{L}_M\Big|_u = -Z\varepsilon^{\mu\nu\kappa\lambda}\mathrm{Trg}\left[\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \gamma_5 \end{array}\right)\left(\begin{array}{c|c} 0 & -\bar{\psi}_\mu\gamma_5 \\ \hline -\gamma_5\psi_\mu & i\Gamma^a B_{\mu a} \end{array}\right)\left(\begin{array}{c|c} 0 & -\bar{\psi}_\nu\gamma_5 \\ \hline -\gamma_5\psi_\nu & i\Gamma^a B_{\nu a} \end{array}\right)\nabla_\kappa\, M\,\nabla_\lambda\, M\right]$$

$$= Z\,\varepsilon^{\mu\nu\kappa\lambda}\Big[m^2\,\mathrm{Tr}(\psi_\mu\bar{\psi}_\nu\gamma_5\nabla_\kappa\chi\overline{\nabla_\lambda\chi}) + m^3\,\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_5\nabla_\kappa\chi\overline{\nabla_\lambda\chi})$$

$$+ \frac{m^2}{4}\nabla_\lambda F^{AB}\,\mathrm{Tr}(\gamma_\mu\gamma_5\psi_\nu\overline{\nabla_\kappa\chi}\,\Sigma_{AB}) + \frac{m}{16}\nabla_\kappa F^{AB}\nabla_\lambda F^{CD}\,\mathrm{Tr}(\psi_\mu\bar{\psi}_\nu\gamma_5\tfrac{1}{2}[\Sigma_{AB},\Sigma_{CD}])$$

$$+ \frac{m^2}{16}\nabla_\kappa F^{AB}\nabla_\lambda F^{CD}\,\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_5\,\tfrac{1}{2}[\Sigma_{AB},\Sigma_{CD}])\Big]$$

(where $\frac{i}{4}F^{AB}\,\Sigma_{AB} = \frac{i}{4}F^{ab}\,\sigma_{ab} - \frac{im}{2}A^a\gamma_a$).

Using the Fierz resummation relation, (3.30) on the spinor bilinears and evaluating the traces we find,

$$\mathcal{L}_M\Big|_u = Z\,\varepsilon^{\mu\nu\kappa\lambda}\Big[\frac{m^2}{16}\varepsilon_{abcd}(\bar{\psi}_\mu\,\sigma^{ab}\,\psi_\nu)(\overline{\nabla_\mu\chi}\,\sigma^{cd}\,\nabla_\lambda\chi)$$

$$- \frac{im^3}{2}\varepsilon_{abcd}\,h_\mu{}^a\,h_\nu{}^b(\overline{\nabla_\kappa\chi}\,\sigma^{cd}\,\nabla_\lambda\chi)$$

$$+ \frac{im^2}{2}h_\mu{}^a(\bar{\psi}_\nu\gamma_b\gamma_5\nabla_\kappa\chi)\nabla_\lambda F_a{}^b + \frac{m^3}{2}h_\mu{}^a(\bar{\psi}_\nu\,\gamma_5\nabla_\kappa\chi)\nabla_\lambda A_a$$

$$- \frac{im^3}{4}\varepsilon_{abcd}\,h_\mu{}^a(\bar{\psi}_\nu\sigma^{bc}\nabla_\kappa\chi)\nabla_\lambda A^d + \frac{im^2}{4}\varepsilon_{abcd}\,h_\mu{}^a(\bar{\psi}_\nu\gamma^b\nabla_\kappa\chi)\nabla_\lambda F^{cd}$$

$$- \frac{im}{8}\varepsilon_{abcd}(\bar{\psi}_\mu\sigma^{ab}\psi_\nu)(m^2\nabla_\kappa A^c\nabla_\lambda A^d + \nabla_\kappa F^{ce}\nabla_\lambda F^d{}_e)$$

$$- m^2\,\varepsilon_{abcd}\,h_\mu{}^a\,h_\nu{}^b(m^2\,\nabla_\kappa A^c\nabla_\lambda A^d + \nabla_\kappa F^{ce}\nabla_\lambda F^d{}_e)\Big] \qquad (7)$$

Our first concern with this Lagrangian is to establish the form it takes in flat space-time, with $\psi_\mu = B_\mu{}^{ab} = m = 0$.

We see immediately that, with $\psi_\mu = 0$,

$$\mathcal{L}_M\Big]_{u,\psi_\mu=0} = Z\,\varepsilon^{\mu\nu\kappa\lambda}\Big[-\frac{im^3}{2}\,\varepsilon_{abcd}\,h_\mu{}^a\,h_\nu{}^b(\overline{\nabla_\kappa\chi}\,\sigma^{cd}\nabla_\lambda\chi)\Big]_{\psi_\mu=0} +$$

$$-\,m^2\,\varepsilon_{abcd}\,h_\mu{}^a\,h_\nu{}^b(m^2\nabla_\kappa A^c\nabla_\lambda A^d + \nabla_\kappa F^{ce}\nabla_\lambda F^d{}_e\Big]_{\psi_\mu=0}\Big]$$

We now wish to carry out the group contraction $m \to 0$, to remove the cosmological term in the gauge action (see Appendix D) and thus ensure that the gravitational vacuum is Minkowski space-time (not anti de Sitter space-time with a constant curvature). Notice however that $\mathcal{L}_M\Big]_{u,\psi_\mu=0} \xrightarrow{m\to 0} 0$, unless $Z = Z(m)$. Now we required $[Z] = (\text{mass})^{-4}$ hence we may take $\underline{Z = m^{-4}}$ then letting $m \to 0$ and setting $B_\mu{}^{ab} = 0$ we obtain the flat space-time Lagrangian,

$$\mathcal{L}_M\Big]_{u,\psi_\mu = B_\mu{}^{ab} = m = 0} =$$

$$\varepsilon^{\mu\nu\kappa\lambda}(-\tfrac{i}{2})\varepsilon_{abcd}\,h_\mu{}^a\,h_\nu{}^b(\tfrac{1}{m}(\partial_\kappa\,\overline{\chi} + \tfrac{im}{2}\,\overline{\chi}\,\gamma_\kappa)\sigma^{cd}(\partial_\lambda\chi - \tfrac{im}{2}\,\gamma_\lambda\chi))\Big]_{m=0}$$

$$+\ \varepsilon^{\mu\nu\kappa\lambda}(-\varepsilon_{abcd})h_\mu{}^a h_\nu{}^b\Big[(\partial_k A^c + F^c{}_e h_k{}^e)(\partial_\lambda A^d + F^d{}_e h_\lambda{}^e) +$$

$$+\ (\tfrac{1}{m^2}\,(\partial_\kappa F^{ce} + m^2(h_\kappa{}^c\,A^e - h_k{}^e A^c))(\partial_\lambda F^d{}_e + m^2(h_\lambda{}^d A_e - h_{\lambda e}A^d)))\Big]_{m=0}\Big]$$

where we have used equations (5) to obtain,

$$\nabla_\mu A^a\Big]_{\psi_\mu = B_\mu{}^{ab}=0} = \partial_\mu A^a + F^a{}_b\,h_\mu{}^b$$

$$\nabla_\mu F^{ab}\Big]_{\psi_\mu = B_m{}^{ab}=0} = \partial_\mu F^{ab} + m^2(h_\mu{}^a\,A^b - h_\mu{}^b A^a)$$

$$\nabla_\mu \chi\Big]_{\psi_\mu = B_\mu{}^{ab}=0} = \partial_\mu \chi - \frac{im}{2}\gamma_\mu \chi \,.$$

In flat space-time it is convenient to work with the global Lorentz coordinates, $x^\mu = \delta^\mu{}_a x^a$. In these coordinate systems, $h_\mu{}^a = \delta_\mu{}^a$ and our Lagrangian becomes,

$$\mathcal{L}_M\Big]_{u,\psi_\mu = B_\mu{}^{ab}=m=0} = \varepsilon^{ab\kappa\lambda}\,\varepsilon_{abcd}\Big[\tfrac{1}{2}\bar{\chi}\,\gamma_\kappa \sigma^{cd}\partial_\lambda \chi + 2(\partial_\kappa A^c)F_\lambda{}^d - F_\kappa{}^c F_\lambda{}^d - 2(\delta_\kappa{}^c A^e - \delta_\kappa{}^e A^c)\partial_\lambda F^d{}_e\Big]$$

where we have dropped the terms,

$$\sim \varepsilon^{ab\kappa\lambda}\,\varepsilon_{abcd}\,\partial_\kappa A^c\,\partial_\lambda A^d$$

and

$$\sim \varepsilon^{ab\kappa\lambda}\,\varepsilon_{abcd}\,\partial_\kappa \bar{\chi}\,\sigma^{cd}\,\partial_\lambda \chi$$

and

$$\sim \varepsilon^{ab\kappa\lambda}\,\varepsilon_{abcd}\,\partial_\kappa F^{ce}\,\partial_\lambda F^d{}_e$$

which are all total divergences.

Now, using,

$$\varepsilon^{ab\kappa\lambda}\,\varepsilon_{abcd} = -2\,\delta^{\kappa\lambda}_{[cd]}$$

and

$$\gamma_c\,\sigma^{cd} = 3i\,\gamma^d$$

we obtain,

$$\mathcal{L}_M\Big]_{u,\psi_\mu = B_\mu{}^{ab}=m=0} = -6i\,\bar{\chi}\,\partial\!\!\!/\,\chi - 12(\partial^a A^b)F_{ab} + 2F^{ab}F_{ab}\,.$$

The F^{db} are auxiliarly fields with the algebraic equations of motion,

$$\frac{\partial \mathcal{L}_M\big]}{\partial F^{ab}} = -6(\partial^a A^b - \partial^b A^a) + 4F^{ab} = 0$$

hence, $$\underline{F^{ab} = \frac{3}{2}(\partial^a A^b - \partial^b A^a)} \quad . \tag{8}$$

Substituting (8) back into $\mathcal{L}_M\big|_u$,

$$\underline{\mathcal{L}_M\big|_{u,\psi_\mu=B_\mu{}^{ab}=m=0} = -6i\,\bar{\chi}\not\partial\chi - \frac{9}{2}(\partial^a A^b - \partial^a A^a)(\partial_a A_b - \partial_b A_a)} . \tag{9}$$

This is the Lagrangian for free massless fields A^a and χ_a which, as we discussed in Chapter II (page 112), form a global supersymmetry multiplet, $(\frac{1}{2},1)$. The supergauge transformations for this multiplet were found to be, (equations 2.31)),

$$\underline{\delta_\varepsilon A^a = i\,\bar{\varepsilon}\,\gamma^a\,\chi} \tag{10a}$$

$$\underline{\delta_\varepsilon \chi = \frac{i}{2} F^{ab}\,\sigma_{ab}\,\varepsilon} \tag{10b}$$

where $F^{ab} = (\partial^a A^b - \partial^b A^a)$ and ε_α are constants.

These transformations may be compared with the gauge transformations,

$$\delta_A M = [A,M] \qquad A \,\varepsilon\, \text{algebra of } Osp(1,4)$$

which in component form are written

$$\underline{\delta_A A^a = \omega^a{}_b A^b - \xi^b F^a{}_b + i\,\bar{\varepsilon}\,\gamma^a\,\chi} \tag{11a}$$

$$\underline{\delta_A F^{ab} = \omega^a{}_c F^{cb} + \omega^b{}_c F^{ac} - m^2(\xi^a A^b - \xi^b A^a) - im\,\bar{\varepsilon}\,\sigma^{ab}\chi} \tag{11b}$$

$$\underline{\delta_A \chi = -\frac{i}{4}\omega^{ab}\sigma_{ab}\chi + \frac{im}{2}\xi^a\gamma_a\chi + \frac{i}{4}F^{ab}\sigma_{ab}\varepsilon - \frac{im}{2}A^a\gamma_a\varepsilon} \quad . \tag{11c}$$

We see that (11a) contains (10a) and (11c), with $m=0$, contains (10b).

In fact, the flat space-time Lagrangian (9) is not invariant under these supergauge transformations since the two kinetic terms are not in the required ratio for global supersymmetry. This is not surprising since we have evaluated $\mathcal{L}_M$ in the unitary gauge and we have a situation analogous to that in Chapter III for the gauge action where we must choose the appropriate linear combination of terms to recover the supersymmetry (though not the translational) invariance in the unitary gauge. With the present problem we find that we may add to $\mathcal{L}_M$, the quadratic action,

$$\underline{\mathcal{L}_2 \;=\; \mathrm{Trg}(M^2) \;=\; \tfrac{1}{2}F^{ab}F_{ab} + m^2A^aA_a - 2m\,\bar{\chi}\chi} \quad . \tag{12}$$

After group contraction $\mathcal{L}_2$ is simply $\frac{1}{2}F^{ab}F_{ab}$ and may be added to the flat space action (9) to restore the invariance under (10). Notice that $\mathcal{L}_2$ shows that the mass terms for the fields are $\sim m = \frac{1}{R}$, just as we found for the gravitino in the previous chapter. Our results for the flat space-time reduction of the adjoint multiplet differ from Chamseddine[8] who obtained fields χ_α and A^a with masses $\sim\frac{1}{\kappa}$.

Now let us consider the vector multiplet $V = \begin{pmatrix} \phi \\ b\chi_\alpha \end{pmatrix}$

where b is a constant, $[b] = (\mathrm{mass})^{-\frac{1}{2}}$.

An invariant action containing the kinetic terms is,

$$\underline{\mathcal{L}_V \;=\; Z\,\varepsilon^{\mu\nu\kappa\lambda}\,\mathrm{Trg}(mH\,\nabla_\mu(mH)\nabla_\mu(mH)\,\nabla_\kappa V\,\overline{\nabla_\lambda V})} \tag{13}$$

where: Z is a constant, $[Z] = (\mathrm{mass})^{-2}$ so that $[\mathcal{L}_V] = (\mathrm{mass})^4$

$$\underline{\nabla_\kappa V \;=\; \partial_\kappa V - \omega_\kappa V \;\equiv\; \begin{pmatrix} \nabla_\kappa\,\phi \\ b\nabla_\kappa\,\chi \end{pmatrix}} \tag{14}$$

so that,

$$\underline{\nabla_\kappa \phi = \partial_\kappa \phi + b\sqrt{m}\ \bar{\psi}_\kappa \chi} \tag{15a}$$

$$\underline{\nabla_\kappa \chi = D_\kappa \chi - \frac{im}{2}\gamma_\kappa \chi - \frac{\sqrt{m}}{b}\phi\,\psi_\kappa}\ . \tag{15b}$$

$$\overline{\nabla_\lambda V} \equiv (\nabla_\lambda V)^T G = (\nabla_\lambda V)^T \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & C \end{array}\right) = (\nabla_\lambda \phi,\ b\ \overline{\nabla_\lambda \chi}\ )\ .$$

Hence

$$\varepsilon^{\mu\nu\kappa\lambda}\ \nabla_\kappa V\ \overline{\nabla_\lambda V} = \varepsilon^{\mu\nu\kappa\lambda} \left(\begin{array}{c|c} \nabla_\kappa\phi\nabla_\lambda\phi & b\nabla_\kappa\phi\ \overline{\nabla_\lambda\chi} \\ \hline b\nabla_\kappa\chi\nabla_\lambda\phi & b^2\nabla_\kappa\chi\ \overline{\nabla_\lambda\chi} \end{array}\right)$$

$$= \frac{b}{2}\varepsilon^{\mu\nu\kappa\lambda} \left(\begin{array}{c|c} 0 & (\nabla_\kappa\phi\ \overline{\nabla_\lambda\chi} - \nabla_\lambda\phi\ \overline{\nabla_\kappa\chi}\ ) \\ \hline (\nabla_\kappa\chi\nabla_\lambda\phi - \nabla_\lambda\chi\nabla_\kappa\phi) & (\nabla_\kappa\chi\ \overline{\nabla_\lambda\chi} - \nabla_\lambda\chi\ \overline{\nabla_\kappa\chi}\ )b \end{array}\right)$$

$$= \frac{b}{2}\varepsilon^{\mu\nu\kappa\lambda} \left(\begin{array}{c|c} 0 & -(\nabla_\kappa\chi\nabla_\lambda\phi - \nabla_\lambda\chi\nabla_\kappa\phi)^T C \\ \hline (\nabla_\kappa\chi\nabla_\lambda\phi - \nabla_\lambda\chi\nabla_\kappa\phi) & \frac{b}{4}(\overline{\nabla_\kappa\chi}\ \Sigma^{AB}\nabla_\lambda\chi)\Sigma_{AB} \end{array}\right)$$

so we recognize that $\varepsilon^{\mu\nu\kappa\lambda}\ \nabla_\kappa V\ \overline{\nabla_\lambda V} \subset$ algebra Osp(1,4).

Unfortunately this does not help with the problem of rescaling the fields by $m = \frac{1}{R}$ since no rescaling, $\phi \to f_1(m)\phi$, $\chi \to f_2(m)\chi$ will introduce m in the correct manner (see equation (4)).

Evaluating $\mathcal{L}_V$ in the unitary gauge we find

$$\underline{\mathcal{L}_V\Big|_u = Z\ \varepsilon^{\mu\nu\kappa\lambda}\Big[-im^{3/2}\ b\ h_\mu{}^a(\bar{\psi}_\nu\gamma_a\gamma_5\ \nabla_\kappa\chi)\nabla_\lambda\phi}$$

$$\underline{-\frac{mb^2}{16}\ \varepsilon_{abcd}(\bar{\psi}_\mu\sigma^{ab}\ \psi_\nu)(\overline{\nabla_\kappa\chi}\ \sigma^{cd}\ \nabla_\lambda\chi)}$$

$$\underline{+\frac{im^2b^2}{2}\ \varepsilon_{abcd}\ h_\mu{}^a\ h_\nu{}^b(\overline{\nabla_\kappa\chi}\ \sigma^{cd}\nabla_\lambda\chi)\Big]}\ . \tag{16}$$

The relevant term for the flat space-time reduction with $\psi_\mu = B_\mu{}^{ab} = m = 0$ is

$$\mathcal{L}_V \Big]_{u,\psi_\mu=0} = \frac{i}{2} Z\, b^2 m^2\, \varepsilon^{\mu\nu\kappa\lambda}\, \varepsilon_{abcd}\, h_\mu{}^a\, h_\nu{}^b (\overline{\nabla_\kappa \chi}\, \sigma^{cd}\, \nabla_\lambda \chi) \Big]_{\psi_\mu=0} .$$

We see that $\mathcal{L}_V$ vanishes after group contraction unless $\underline{Z = Z(m) = m^{-2}}$. With $Z = m^{-2}$ we set $B_\mu{}^{ab} = 0$ and $m = 0$ and find,

$$\mathcal{L}_V \Big]_{u,\psi_\mu=B_\mu{}^{ab}=m=0} =$$

$$\frac{i}{2} b^2\, \varepsilon^{\mu\nu\kappa\lambda}{}_{abcd}\, h_\mu{}^a\, h_\nu{}^b (\partial_\kappa \bar{\chi} + \frac{im}{2} \bar{\chi}\gamma_\kappa)\sigma^{cd}(\partial_\lambda \chi - \frac{im}{2}\gamma_\lambda \chi)\Big]_{m=0} .$$

Working in Lorentz coordinates $\chi_\mu = \delta_\mu{}^a \chi_a$, so that $h_\mu{}^a = \delta_\mu{}^a$ and using $\varepsilon^{\mu\nu\kappa\lambda}\, \varepsilon_{\mu\nu cd} = -2\delta^{\kappa\lambda}_{[cd]}$, we find

$$\mathcal{L}_V \Big]_{u,\psi_\mu=B_\mu{}^{ab}=m=0} =$$

$$- i\, b^2\, \delta^{ab}_{[cd]} (\partial_a \bar{\chi} + \frac{im}{2} \bar{\chi}\gamma_a)\sigma^{cd}(\partial_b \chi - \frac{im}{2}\gamma_b \chi)\Big]_{m=0} .$$

This action is simply the total divergence,

$$i\, b^2\, \delta^{ab}_{[cd]}\, \partial_a \bar{\chi}\, \sigma^{cd}\, \partial_b \chi = i\, b^2\, \delta^{ab}_{[cd]}\, \partial_a (\bar{\chi}\, \sigma^{cd}\, \partial_b \chi)$$

unless $b = b(m) = m^{-\frac{1}{2}}$, in which case

$$\mathcal{L}_V \Big]_{u,\psi_\mu=B_\mu{}^{ab}=m=0} = \delta^{ab}_{[cd]}\, \bar{\chi}\, \gamma_a\, \sigma^{cd}\, \partial_b \chi = 6i\, \bar{\chi} \not{\partial} \chi .$$

Hence our action, (13), for the vector multiplet reduces to a total divergence in the flat-space time limit unless we arbitrarily rescale the spinor field χ_α in V by $b = m^{-\frac{1}{2}}$. Even with this rescaling we only recover a kinetic term for the spinor field χ_α and it is clear that any action formed with the $\varepsilon^{\mu\nu\kappa\lambda}$ will not yield kinetic terms, $\sim \partial_\mu\phi\, \partial_\mu\phi$ for the scalar field. There are also problems with possible mass terms for the fields which we see simply by evaluating,

$$\mathrm{Trg}\, V\,\overline{V} \;=\; \overline{V}\,V \;=\; \phi^2 + b^2\,\overline{\chi}\,\chi .$$

Multiplying this through by m^2 we obtain masses $\sim m = \frac{1}{R}$ only if $b = m^{-\frac{1}{2}}$. The vector multiplet fails to yield a flat space-time supermultiplet. In Chapter II we discussed the supermultiplet consisting of one spinor and two scalar fields (the Wess-Zumino multiplet) and saw that there was no multiplet consisting of one spinor and one scalar field. We therefore dismiss the vector multiplet as unphysical, at least in flat space-time, and henceforth limit our attention to the adjoint multiplet M.

2) Coupling the Adjoint Multiplet to Supergravity

The locally supersymmetric coupling of the massless spin 1 multiplet (A^a, χ_α) to supergravity was obtained by Ferrara et al.[(26),(27)], using the Noether coupling prescription. We shall briefly review their results and then see how our action, equation (7), compares in the group contraction limit. The 'Noether method' is a rather inelegant approach to gauging global symmetries, used when the group structure of these symmetries is not known. First consider the action $\int \mathcal{L}_0 dx$ for the multiplet (A_a, χ_α) where

$$\mathcal{L}_0 \;=\; -\frac{i}{4} F^{ab} F_{ab} - \frac{i}{2}\overline{\chi}\,\partial\!\!\!/\,\chi, \qquad F^{ab} = (\partial^a A^b - \partial^b A^a) \qquad (17)$$

is the free field Lagrangian for (A_a, χ_α). In Chapter II (page 113) we found that, under the supergauge transformations,

$$\delta_\varepsilon A^a = i\,\bar{\varepsilon}\,\gamma^a\,\chi \tag{18a}$$

$$\delta_\varepsilon \chi = \frac{i}{2} F^{ab}\,\sigma_{ab}\,\varepsilon \tag{18b}$$

with constant spinorial parameters ε_α, the Lagrangian $\mathcal{L}_o$ transforms by a total divergence (equation (2.32))

$$\delta_\varepsilon \mathcal{L}_o = \partial_a\left[-i\,\bar{\varepsilon}\gamma_b\chi\,F^{ab} - \frac{1}{4}\bar{\varepsilon}\,\sigma^{bc}\gamma^a\chi\,F_{bc}\right] \equiv \partial_a K^a . \tag{19}$$

Now, in general for a Lagrangian $\mathcal{L}_o = \mathcal{L}_o(\Phi, \partial_a\Phi)$ then,

$$\delta\mathcal{L}_o = \frac{\partial\mathcal{L}_o}{\partial\Phi}\,\delta\Phi + \frac{\partial\mathcal{L}_o}{\partial\Phi,a}\,\delta\Phi,a \qquad (\Phi,a \equiv \partial_a\Phi)$$

then provided $\delta\Phi,a = \partial_a\,\delta\Phi$

we see that

$$\delta\mathcal{L}_o = \partial_a\left(\frac{\partial\mathcal{L}_o}{\partial\Phi,a}\,\delta\Phi\right) + \left(\frac{\partial\mathcal{L}_o}{\partial\Phi} - \partial_a\,\frac{\partial\mathcal{L}_o}{\partial\Phi,a}\right)\delta\Phi .$$

Hence equation (19) may be written,

$$\delta_\varepsilon\mathcal{L}_o = \partial_a\left(\frac{\partial\mathcal{L}_o}{\partial\Phi,a}\,\delta_\varepsilon\Phi\right) + \left(\frac{\partial\mathcal{L}_o}{\partial\Phi} - \partial_a\left(\frac{\partial\mathcal{L}_o}{\partial\Phi,a}\right)\right)\delta_\varepsilon\Phi = \partial_a K^a .$$

The Noether current, $N^a{}_\alpha$ is defined by

$$\underline{\bar{\varepsilon}\,N^a = \frac{\partial\mathcal{L}_o}{\partial\Phi,a}\,\delta_\varepsilon\Phi - K^a} \tag{20}$$

and we see that $N^a{}_\alpha$ is conserved on-shell, (i.e. $\partial_a N^a{}_\alpha = 0$, by definition). From equations (17) and (18) we calculate,

$$\frac{\partial\mathcal{L}_o}{\partial\Phi,a}\,\delta_\varepsilon\Phi = -i\,\bar{\varepsilon}\,\gamma_b\chi\,F_{ab} + \frac{i}{4}\bar{\varepsilon}\,\sigma_{bc}\gamma^a\chi\,F^{bc} .$$

Substituting this and the expression for K^a into (20) we see that,

$$\underline{N^a_\alpha = \frac{1}{2}\sigma_{bc}\gamma^a\chi F^{bc}} \quad . \tag{21}$$

Now if we let the spinorial parameters ε_α in the transformations (18) be locally valued, $\varepsilon_\alpha = \varepsilon_\alpha(x)$ then the variation of $\mathcal{L}_o$ under these transformations becomes,

$$\delta_\varepsilon \mathcal{L}_o = \partial_a K^a - (\partial_a \bar{\varepsilon}^\alpha) K^a_\alpha + \frac{\partial \mathcal{L}_o}{\partial \Phi_{,a}} \delta_{\varepsilon,a} \Phi$$

(where $K^a \equiv \bar{\varepsilon}^\alpha K^a_\alpha$) .

Comparing this with (20) we recognize that

$$\underline{\delta_\varepsilon \mathcal{L}_o = \partial_a K^a + (\partial_a \bar{\varepsilon}^\alpha) N^a_\alpha} \quad . \tag{22}$$

Hence $\mathcal{L}_o$ no longer transforms as a total divergence under the local supersymmetry transformations (18) and must be modified by the introduction of gauge fields with indices determined by the $\partial_a \bar{\varepsilon}^\alpha$ term they must cancel. We see that the second term in (22) is cancelled by introducing the gauge field $\psi_a{}^\alpha$ through the Noether coupling term $-\kappa \bar{\psi}_a N^a$ (κ is the gravitational coupling constant) and requiring $\psi_{\mu\alpha}$ to transform as,

$$\underline{\delta_\varepsilon \psi_{\mu\alpha} = \frac{1}{\kappa}\partial_a \varepsilon_\alpha + \ldots\ldots} \quad . \tag{23}$$

The Lagrangian $\mathcal{L}_o$ has now become, (using (21))

$$\underline{\mathcal{L} = -\frac{i}{4}(F_{ab})^2 - \frac{i}{2}\bar{\chi}\not{\partial}\chi - \frac{\kappa}{2}\bar{\psi}_a \sigma_{bc}\gamma^a\chi F^{bc}} \tag{24}$$

and by construction,

$$\delta_\varepsilon \mathcal{L} = \partial_a K^a + O(\kappa) \ .$$

The action is invariant up to terms of order κ. These terms which

which break the invariance at $O(\kappa)$ arise from the variations (18) of A^a and χ_α in the Noether coupling term. The Lagrangian (24) is globally Lorentz invariant and we may gauge this symmetry according to the prescription established in the first chapter. The Lorentz field strengths $R_{\mu\nu}{}^{ab}$ are related to the Riemann curvature tensor by $R_{\mu\nu}{}^{ab}\, h_{\kappa a}\, h_{\lambda b} = R_{\mu\nu\kappa\lambda}$ and so we are generally working in curved space-time with world coordinates x^μ. The appropriate Lorentz covariant form of (24) is therefore,

$$\mathcal{L} = \sqrt{-g}\,\{-\tfrac{1}{4}\, g^{\mu\kappa}\, g^{\nu\lambda}\, F_{\mu\nu}\, F_{\kappa\lambda} - \tfrac{i}{2}\,\bar{\chi}\, \not{D}\, \chi - \tfrac{\kappa}{2}\,\bar{\psi}_\mu\, \sigma_{\nu\kappa}\, \gamma^\mu \chi\, F^{\nu\kappa}\}$$

and we supplement the transformations (18) with the supergravity transformations (2.47),

$$\underline{\delta_\varepsilon h_\mu{}^a = i\,\kappa\,\bar{\varepsilon}\,\gamma^a\,\psi_\mu} \tag{25a}$$

$$\underline{\delta_\varepsilon \psi_\mu = \frac{1}{\kappa} D_\mu\, \varepsilon = \frac{1}{\kappa}\,(\partial_\mu \varepsilon + \frac{i}{4}\, B_\mu{}^{ab} \sigma_{ab}\, \varepsilon)} \tag{25b}$$

where (25b) is a modification of (23).

Since $\delta_\varepsilon h_\mu{}^a \sim O(\kappa)$ it is easy to see that $\mathcal{L}$ still transforms only by a total divergence up to $O(\kappa)$. To recover invariance at $O(\kappa)$ further modifications to $\mathcal{L}$ and to the transformations (18) and (25) are required. These were spotted by Ferrara et al.[(26),(27)] who found that the $O(\kappa^2)$ terms were then automatically invariant. The 'first order' form of their invariant action is,

$$\underline{\mathcal{L} = \sqrt{-g}\,\Big[-\tfrac{1}{4} g^{\mu\kappa}\, g^{\nu\lambda}\, F_{\mu\nu}\, F_{\kappa\lambda} - \tfrac{i}{2}\,\bar{\chi}\, \not{D}\, \chi - \tfrac{\kappa}{2}\,(\bar{\psi}_\mu \sigma^{\nu\kappa} \gamma^\mu \chi) F_{\nu\kappa}}$$

$$\underline{- \tfrac{i}{2}\,\kappa^2 (\bar{\psi}_\mu \sigma^{\nu\kappa} \gamma^\mu \chi)(\bar{\psi}_\nu \gamma_\kappa\, \chi)\Big]} \quad . \tag{26}$$

The modified supergauge transformations are,

$$\underline{\delta_\varepsilon A^\mu \quad = \quad i\,\bar{\varepsilon}\,\gamma^\mu \chi} \tag{27a}$$

$$\underline{\delta_\varepsilon \chi \quad = \quad \frac{i}{2}\,(F^{\mu\nu} + \kappa\,\bar{\psi}^\mu\,\gamma^\nu \chi)\sigma_{\mu\nu}\,\varepsilon} \tag{27b}$$

$$\underline{\delta_\varepsilon h_\mu{}^a \quad = \quad i\,\kappa\,\bar{\varepsilon}\,\gamma^a \psi_\mu} \tag{27c}$$

$$\underline{\delta_\varepsilon \psi_\mu \quad = \quad \frac{1}{\kappa}\,D_\mu \varepsilon + \frac{\kappa}{4}(\bar{\chi}\gamma^\nu\gamma_5\chi)\gamma_\nu\gamma_5\gamma_\mu\varepsilon} \quad . \tag{27d}$$

One unsatisfactory feature of these transformations is the occurrence of the matter field χ_α in the transformation (27d) for the gauge field ψ_μ. This may be overcome by the introduction of the six auxiliary fields of the supergravity gauge multiplet (see Chapter II, page 133), four of which form a pseudo-vector V_μ. This V_μ is then introduced into (26) in such a way that its algebraic field equation yields $V_\mu \sim \kappa\,\bar{\chi}\,\gamma_\mu\gamma_5\chi$ which may be substituted into (27d).

Since the Lagrangian $\mathcal{L}$ in equation (26) is quadratic in the matter fields we might expect to identify it within our Osp(1,4) adjoint multiplet action (7). We may substitute for $Z = m^{-4}$ (obtained from the flat space-time reduction analysis) into (7) and rewrite it here as

$$\begin{aligned}
\mathcal{L}_{\underline{M}}\Big|_u \quad = \quad & \varepsilon^{\mu\nu\kappa\lambda}\left[\frac{i}{2m^2}\,h_\mu{}^a(\bar{\psi}_\nu\gamma^b\gamma_5\nabla_\kappa\chi)\nabla_\lambda F_{ab} + \frac{1}{2m}\,h_\mu{}^a(\bar{\psi}_\nu\;\gamma_5\nabla_\kappa\chi)\nabla_\lambda A_a\right] \\
& + \varepsilon^{\mu\nu\kappa\lambda}\;\varepsilon_{abcd}\left[\frac{1}{16m^2}(\bar{\psi}_\mu\sigma^{ab}\psi_\nu)\,(\overline{\nabla_\kappa\chi}\;\sigma^{cd}\nabla_\lambda\chi)\right. \\
& - \frac{i}{2m}\,h_\mu{}^a h_\nu{}^b(\overline{\nabla_\kappa\chi}\;\sigma^{cd}\nabla_\lambda\chi) - \frac{i}{4m}\,h_\mu{}^a(\bar{\psi}_\nu\sigma^{bc}\nabla_\kappa\chi)\nabla_\lambda A^d \\
& + \frac{i}{4m^2}\,h_\mu{}^a(\bar{\psi}_\nu\gamma^b\nabla_\kappa\chi)\nabla_\lambda F^{cd} - \frac{i}{8m}(\bar{\psi}_\mu\sigma^{ab}\psi_\nu)\nabla_\kappa A^c\nabla_\lambda A^d \\
& \left. - \frac{i}{8m^3}\,(\bar{\psi}_\mu\sigma^{ab}\psi_\nu)\nabla_\kappa F^{ce}\nabla_\lambda F^d{}_e - h_\mu{}^a\,h_\nu{}^b\nabla_\kappa A^c\nabla_\lambda A^d - \frac{1}{m^2}h_\mu{}^a h_\nu{}^b\nabla_\kappa F^{ce}\nabla_\lambda F^d{}_e\right]
\end{aligned} \tag{28}$$

where, (equations (5))

$$\nabla_\mu A^a = D_\mu A^a + F^a{}_b\, h_\mu{}^b - i\,\bar{\psi}_\mu \gamma^a \chi$$

$$\nabla_\mu F^{ab} = D_\mu F^{ab} + m^2(h_\mu{}^a A^b - h_\mu{}^b A^a) + im\,\bar{\psi}_\mu \sigma^{ab}\chi$$

$$\nabla_\mu \chi = D_\mu \chi - \frac{im}{2}\gamma_\mu \chi - \frac{i}{4} F^{ab}\sigma_{ab}\psi_\mu + \frac{im}{2} A^a \gamma_a \psi_\mu \; .$$

By inspection, we see that $\mathcal{L}_{\underline{M}}\big|_u$ may be written as,

$$\mathcal{L}_{\underline{M}}\big|_u = \frac{1}{m^3}\mathcal{L}_{(-3)} + \frac{1}{m^2}\mathcal{L}_{(-2)} + \frac{1}{m}\mathcal{L}_{(-1)} + \mathcal{L}_{(0)} + m\,\mathcal{L}_{(1)} + m^2\mathcal{L}_{(2)} \tag{29}$$

where we have simply expanded $\mathcal{L}_{\underline{M}}\big|_u$ in the various powers of m. It is the term, $\mathcal{L}_{(0)}$, independent of m which we can compare with (26). From (28) we identify $\mathcal{L}_{(0)}$ and after a little simplification and rearrangement write it as

$$\begin{aligned}
\mathcal{L}_{(0)} = {} & \epsilon^{\mu\nu\kappa\lambda}\,\epsilon_{abcd}\Big[-\tfrac{1}{2}\, h_\mu{}^a h_\nu{}^b\, \bar{\chi}\gamma_\kappa \sigma^{cd} D_\lambda \chi \\
& + \frac{i}{8} h_\mu{}^a h_\nu{}^b\, \bar{\chi}\gamma_\kappa \sigma^{cd}\sigma_{mn}\psi_\lambda F^{mn} \\
& + h_\mu{}^a h_\nu{}^b F^c{}_e\, h_\kappa{}^e F^d{}_f\, h_\lambda{}^f\Big] \\
& + \epsilon^{\mu\nu\kappa\lambda}\epsilon_{abcd}\Big[\frac{1}{64}(\bar{\psi}_\mu\sigma^{ab}\psi_\nu)(i\bar{\chi}\gamma_\kappa - \bar{\psi}_\kappa A)\sigma^{cd}(-i\gamma_\lambda\chi + A\psi_\lambda) \\
& - \frac{i}{2} h_\mu{}^a h_\nu{}^b\,\bar{\psi}_\kappa A\,\sigma^{cd}(D_\lambda\chi - \tfrac{1}{2}F\psi_\lambda) \\
& - \frac{i}{8} h_\mu{}^a\,\bar{\psi}_\nu\sigma^{bc}(-i\gamma_\kappa\chi + A\psi_\kappa)(D_\lambda A^d - i\bar{\psi}_\lambda\gamma^d\chi) \\
& + \frac{i}{4} h_\mu{}^a\,\bar{\psi}_\nu\gamma^b(D_\kappa\chi - \tfrac{1}{2}F\psi_\kappa)(h_\lambda{}^c A^d - h_\lambda{}^d A^c) \\
& - \frac{1}{8} h_\mu{}^a\,\bar{\psi}_\nu\gamma^b(-i\gamma_\kappa\chi + A\psi_\kappa)(\bar{\psi}_\lambda\sigma^{cd}\chi)
\end{aligned}$$

$$+\frac{1}{4}(\bar{\psi}_\mu\ \sigma^{ab}\psi_\nu)(h_\kappa{}^c\ A^e - h_\kappa{}^e\ A^c)(\bar{\psi}_\lambda\ \sigma^d{}_e\chi)$$

$$+ h_\mu{}^a h_\nu{}^b D_\kappa A^c(D_\lambda A^d + 2F^d{}_e h_\lambda{}^e - 2i\bar{\psi}_\lambda\gamma^d\chi) + h_\mu{}^a h_\nu{}^b\ \bar{\psi}_\kappa\ \gamma^c\chi\ \bar{\psi}_\lambda\ \gamma^d\chi$$

$$-2h_\mu{}^a h_\nu{}^b(D_\kappa F^{ce})(h_\lambda{}^d A_e - h_{\lambda e}A^d) + h_\mu{}^a h_\nu{}^b\bar{\psi}_\kappa\ \sigma^{ce}\chi\ \bar{\psi}_\lambda\ \sigma^d{}_e\chi\ \Big]$$

$$+ \varepsilon^{\mu\nu\kappa\lambda}\ h_\mu{}^a\Big[-\frac{i}{2}\ \bar{\psi}_\nu\gamma_b\gamma_5(D_\kappa x - \frac{1}{2}F\psi_\kappa)(h_\lambda{}^a A^b - h_\lambda{}^b A^a)$$

$$-\frac{i}{4}\ \bar{\psi}_\nu\gamma_b\gamma_5(\gamma_\kappa\chi + iA\psi_\kappa)(\bar{\psi}_\lambda\ \sigma^{ab}\ \chi)$$

$$-\frac{i}{4}(\bar{\psi}_\nu\gamma_5\gamma_\kappa\chi)(D_\lambda A_a + F_{ab}\ h_\lambda{}^b - i\ \bar{\psi}_\lambda\gamma_a\chi)\ \Big]\ . \tag{30}$$

$$(A \equiv i\ A^a\gamma_a, \quad F \equiv \frac{i}{2}\ F^{db}\sigma_{ab})$$

The first three terms of $\mathcal{L}_{(0)}$ occur in the Lagrangian (26):

The first term is $-\frac{1}{2}\ \varepsilon^{\mu\nu\kappa\lambda}\ \varepsilon_{abcd}\ h_\mu{}^a\ h_\nu{}^b\ \bar{\chi}\ \gamma_\kappa\ \sigma^{cd}\ D_\lambda\chi$

but $\qquad \varepsilon^{\mu\nu\kappa\lambda}\ \varepsilon_{abcd}\ h_\mu{}^a\ h_\nu{}^b = 2h\ h^\kappa{}_{[c}\ h^\lambda{}_{d]}$

(Appendix C)

so that first term becomes $\qquad - 2h\ \bar{\chi}\ \gamma_\kappa\ \sigma^{\kappa\lambda}\ D_\lambda\chi$

$$= \underline{-6i\ h\ \bar{\chi}\ \gamma^\lambda D_\lambda\chi} \equiv \underline{-6i\ h\ \bar{\chi}\ \not{D}\ \chi}\ .$$

The second term is $\frac{i}{8}\ \varepsilon^{\mu\nu\kappa\lambda}\ \varepsilon_{abcd}\ h_\mu{}^a\ h_\nu{}^b\ \bar{\chi}\ \gamma_\kappa\ \sigma^{cd}\ \sigma_{mn}\ \psi_\lambda\ F^{mn}$

$$= \frac{i}{2}h\ \bar{\chi}\ \gamma_\kappa\ \sigma^{\kappa\lambda}\sigma_{mn}\ \psi_\lambda\ F_{mn}$$

$$= -\frac{3}{2}h\ \bar{\chi}\ \gamma^\lambda\ \sigma_{mn}\ \psi_\lambda\ F_{mn}$$

$$= \underline{-\frac{3}{2}h\ \bar{\psi}_\lambda\ \sigma_{mn}\ \gamma^\lambda\ \chi\ F^{mn}}$$

The third term is $\qquad \varepsilon^{\mu\nu\kappa\lambda}\ \varepsilon_{abcd}\ h_\mu{}^a\ h_\nu{}^b\ F^c{}_e\ h_\kappa{}^e\ F^d{}_f\ h_\lambda{}^f$

$$= 2h\ h^\kappa{}_{[c}\ h^\lambda{}_{d]}\ F^c{}_\kappa F^d{}_\lambda$$

$$= \underline{-2h(F_{\kappa\lambda})^2}\ .$$

The last term in (26), quartic in the spinors is identified in the remaining terms in $\mathcal{L}_{(0)}$ when we set $A^a = 0$. The major problem with $\mathcal{L}_{(0)}$ is the existence of the terms containing $A^d(x)$. In the Lagrangian, (26), A^a only occurs through the $F^{ab} = \partial^a A^b - \partial^b A^a$ so that in the absence of spinor fields the Lagrangian (26) reduces to $-\frac{1}{4}\sqrt{-g}\,(F_{\mu\nu})^2$ which is the Maxwell action in curved space-time. In $\mathcal{L}_{(0)}$ the removal of the spinor fields leaves, (recall that torsion vanishes in the absence of spin)

$$\begin{aligned}
\mathcal{L}_{\underline{o}}\Big|_{\psi_\mu=\chi=0} &= -2\sqrt{-g}\,(F_{\mu\nu})^2 + 4\sqrt{-g}\,F^{\kappa d}(D_\kappa A_d) + 2\sqrt{-g}\,h^\kappa{}_{[c}\,h^\lambda{}_{d]}D_\kappa A^c D_\lambda A^d \\
&= -\frac{2}{3}\sqrt{-g}\,(F_{\mu\nu})^2 + 2\sqrt{-g}\,h^\kappa{}_{[c}\,h^\lambda{}_{d]}\,D_\kappa\,A^c\,D_\lambda\,A^d \\
&= -\frac{2}{3}\sqrt{-g}\,(F_{\mu\nu})^2 + \sqrt{-g}\,h^\mu{}_{[a}\,h^\nu{}_{b]}\,A^a\,R_{\mu\nu}{}^{bc}A_c \\
&= \underline{-\frac{2}{3}\sqrt{-g}\,(F_{\mu\nu})^2 - 2\sqrt{-g}\,R_{\mu\nu}\,A^\mu A^\nu} \ .
\end{aligned}$$

The second term in $\mathcal{L}_{\underline{o}}\Big|_{\psi_\mu=\chi=0}$ is a non-minimal coupling of the Maxwell field A^μ to gravity.

We must remember that $\mathcal{L}_{(0)}$ is part of an Osp(1,4) action evaluated in the unitary gauge. The complicated form of (30), largely due to the terms containing A^a explicitly, makes the problem of identifying the local supersymmetry transformations, under which $\mathcal{L}_o$ is invariant, extremely difficult. This problem is intensified when we consider the requirement that the group contraction limit of $\mathcal{L}_{\underline{M}}\Big|_u$ is just $\mathcal{L}_{(0)}$, which should be the case from the definition of $\mathcal{L}_{(0)}$ in (29). As was the case for the gauge actions in Chapter III, we require the terms with the factors $\frac{1}{m}$, $\frac{1}{m^2}$, $\frac{1}{m^3}$ to be topological invariants in order to enable us to set m=0. Therefore $\mathcal{L}_{(-3)}$, $\mathcal{L}_{(-2)}$ and $\mathcal{L}_{(-1)}$ must each be the integrand of a topological invariant. From (28) we identify the simplest of these, $\mathcal{L}_{(-3)}$ as

$$\underline{\mathcal{L}_{(-3)} = -\frac{i}{8}\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}(\bar{\psi}_\mu\sigma^{ab}\psi_\nu)D_\kappa F^{ce}D_\lambda F^d{}_e} \tag{31}$$

so that $\mathcal{L}_{(-3)} = \mathcal{L}_{(-3)}(\psi_\mu, B_\mu{}^{ab}, F^{db})$ and we calculate

$$\frac{\delta I_{(-3)}}{\delta\bar{\psi}_\mu{}^\alpha} = -\frac{i}{4}\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}(\sigma^{ab}\psi_\nu)_\alpha\, D_\kappa F^{ce}\, D_\lambda F^d{}_e \tag{32a}$$

$$\frac{\delta I_{(-3)}}{\delta B_\kappa{}^{sr}} = \frac{i}{8}\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}(\bar{\psi}_\mu\sigma^{ab}\psi_\nu)(\delta_{[s}{}^c F_{r]}{}^e - \delta_{[s}{}^e F_{r]}{}^c)D_\lambda F^d{}_e \tag{32b}$$

$$\frac{\delta I_{(-3)}}{\delta F^{de}} = -\frac{i}{8}\,\varepsilon^{\mu\nu\kappa\lambda}\left[\varepsilon_{abcd}\, D_\kappa(\bar{\psi}_\mu\sigma^{ab}\psi_\nu\, D_\lambda F^c{}_e) - (e \leftrightarrow d)\right] \tag{32c}$$

Hence for $I_{(-3)} \equiv \int \mathcal{L}_{(-3)}\, d^4x$ to be a topological invariant we require equations (32) to vanish identically, which is clearly <u>not</u> the case. Our only escape would be to write down another OSp(1,4) adjoint multiplet $\mathcal{L}_M$, with a $\frac{1}{m^3}\mathcal{L}_{(-3)}$ term, in the unitary gauge, which cancels (31). However, there appears to be no OSp(1,4) action other than (6) which will contain $\mathcal{L}_{(-3)}$. The situation becomes worse when we consider $\mathcal{L}_{(-2)}$ and $\mathcal{L}_{(-1)}$ where, for example,

$$\begin{aligned}\mathcal{L}_{(-2)} = {} & \frac{i}{2}\,\varepsilon^{\mu\nu\kappa\lambda}\, h_\mu{}^a\, \bar{\psi}_\nu\, \gamma^b\gamma_5(D_\kappa\chi - \tfrac{1}{2}F\psi_\kappa)D_\lambda F_{ab} \\ & + \varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\Big[\frac{1}{16}(\bar{\psi}_\mu\,\sigma^{ab}\psi_\nu)(\overline{D_\kappa\chi} + \tfrac{1}{2}\bar{\psi}_\kappa F)\sigma^{cd}(D_\lambda\chi - \tfrac{1}{2}F\psi_\lambda) \\ & + \frac{i}{4}\, h_\mu{}^a\bar{\psi}_\nu\gamma^b(D_\kappa\chi - \tfrac{1}{2}F\psi_\lambda)D_\lambda F^{cd} + \frac{1}{4}(\bar{\psi}_\mu\sigma^{ab}\psi_\nu)(D_\kappa F^{ce})(\bar{\psi}_\lambda\sigma^d{}_e\chi) \\ & - h_\mu{}^a\, h_\nu{}^b(D_\kappa F^{ce})(D_\lambda F^d{}_e)\Big]\,.\end{aligned}$$

We readily establish that $\frac{\delta I_{-2}}{\delta \bar{\psi}_{\mu}{}^{\alpha}}$ does not vanish so that we require another action $\mathcal{L}_{M''}\big]_{u}$ to cancel $\frac{1}{m^2}\mathcal{L}_{(-2)}$. The difficulty which we have with the $\frac{1}{m^3}\mathcal{L}_{(-3)}$, $\frac{1}{m^2}\mathcal{L}_{(-2)}$ and $\frac{1}{m}\mathcal{L}_{(-1)}$ terms indicates that we cannot take the group contraction limit of the matter action (unless we remove the gauge fields) and we therefore suspend any further analysis of $\mathcal{L}_{(0)}$.

3) Non-Linear Matter Actions

We continue to work with the adjoint matter multiplet M which we now write as

$$\underline{M = \frac{i}{2} F^{ab} M_{ab} + im\, A^{a} M_{5a} + \sqrt{m}\, \bar{\chi}^{\alpha} Q_{\alpha}.} \tag{33}$$

under g Osp(1,4), M transforms as

$$\underline{M \xrightarrow{g} M' = g\, M\, g^{-1}} \quad . \tag{34}$$

This transformation may be calculated explicitly[36] by setting $g = \exp A$, where

$$A = \frac{i}{2}\omega^{ab} M_{ab} + im\, {}^{a}M_{5a} + \sqrt{m}\, \bar{\varepsilon}^{\alpha} Q_{\alpha}.$$

With reference to the previous chapter and Appendix D, the non-linear fields $\hat{M}$ are defined by, (equation (3.60a))

$$\underline{\hat{M} = C^{-1}\, M\, C} \tag{35}$$

where $C = e^{\bar{\lambda}\boldsymbol{Q}}\, e^{iy^{a}M_{5a}} \;\varepsilon\; \frac{Osp(1,4)}{SL(2,c)}$.

The $\hat{M}$ transform under $g \;\varepsilon\; Osp(1,4)$ as

$$\underline{\hat{M} \xrightarrow{g} \hat{M}' = h_1\, \hat{M}\, h_1^{-1}} \tag{36}$$

where $h_1 = h_1(g, \lambda_\alpha, y_a) \in SL(2,c)$.

Non-linear covariant derivatives are formed with the non-linear gauge potentials, (equation (3.61a))

$$\underline{\hat{\omega}_\mu = \frac{i}{2}\hat{B}_\mu{}^{ab} M_{ab} + im\,\hat{h}_\mu{}^a M_{5a} + \sqrt{m}\,\hat{\bar{\psi}}_\mu{}^\alpha Q_\alpha} \tag{37}$$

In fact $\hat{h}_\mu{}^a$ and $\hat{\psi}_{\mu\alpha}$ transform homogeneously, according to equations (3.62a,b), so that we form a non-linear covariant derivative,

$$\underline{\hat{D}_\mu = \partial_\mu - \frac{i}{2}\hat{B}_\mu{}^{ab} M_{ab} \equiv \partial_\mu - \hat{B}_\mu\,.} \tag{38}$$

Then we may directly verify that

$$\underline{\hat{D}_\mu\hat{M} = \partial_\mu \hat{M} - \left[\hat{B}_\mu, M\right]} \tag{39}$$

transforms as $\hat{M}$, i.e.

$$\underline{\hat{D}_\mu\hat{M} \xrightarrow{g} (\hat{D}_\mu\hat{M})' = h_1\,\hat{D}_\mu\,\hat{M}\,h_1{}^{-1}\,.} \tag{40}$$

The components of $\hat{D}_\mu\hat{M}$ are the covariants,

$$\underline{\hat{D}_\mu\hat{A}^a = \partial_\mu\hat{A}^a - \hat{B}_\mu{}^a{}_b\,\hat{A}^b} \tag{41a}$$

$$\underline{\hat{D}_\mu\hat{F}^{ab} = \partial_\mu\hat{F}^{ab} - \hat{B}_\mu{}^a{}_c\,\hat{F}^{cb} - \hat{B}_\mu{}^b{}_c\,\hat{F}^{ac}} \tag{41b}$$

$$\underline{\hat{D}_\mu\hat{\chi}_\alpha = \partial_\mu\,\hat{\chi}_\alpha + \frac{i}{4}\,\hat{B}_\mu{}^{ab}(\sigma_{ab}\chi)_\alpha\,.} \tag{41c}$$

We may now write down non-linear actions for the adjoint multiplet coupled to supergravity by forming linear SL(2,c) invariants from the covariants, $\hat{A}^a$, $\hat{F}^{ab}$, $\hat{\chi}_\alpha$, $\hat{D}_\mu\hat{A}^a$, $\hat{D}_\mu\hat{F}^{ab}$, $\hat{D}_\mu\,\hat{\chi}_\alpha$, $\hat{h}_\mu{}^a$ and $\hat{\psi}_{\mu\alpha}$. Choosing just one example, we may write,

$$\hat{\mathcal{L}}_{(-3)} = \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{abcd} (\bar{\hat{\psi}}_\mu \sigma^{ab} \hat{\psi}_\nu)(\hat{D}_\kappa \hat{F}^{ce})(\hat{D}_\lambda \hat{F}^d{}_e) .$$

This action is invariant under the full non-linear action of OSp(1,4). In the unitary gauge,

$$\hat{\mathcal{L}}_{(-3)} \longrightarrow \mathcal{L}_{(-3)} = \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{abcd} (\bar{\psi}_\mu \sigma_{ab} \psi_\nu) D_\kappa F^{ce} D_\lambda F^d{}_e$$

and $\mathcal{L}_{(-3)}$ is invariant only under the SL(2,c) subgroup. We chose as our example the $\frac{1}{m^3}$ term (31) in the constrained Higgs field Lagrangian. It is clear that we may similarly construct any term in $\mathcal{L}_{\underline{M}}\big|_u$ of equations (28) and in particular we could reconstruct the Lagrangian (26) of Ferrara et al., invariant under the supersymmetry transformations (27). There are in fact so few restrictions on the non-linear actions which can be written down that this method appears to contribute very little to the problem of constructing supersymmetric couplings of matter to gravity.

Conclusions

In this chapter we set out to explore the possible guidance that the spontaneously broken gauge theory of OSp(1,4) could afford to the problem of constructing supersymmetric matter couplings to gravity. Any theory of matter coupling to OSp(1,4) supergravity is based on a matter multiplet which forms an irreducible representation of Osp(1,4). Our analysis (in particular of the dimensions of fields) leads us to propose that it may only be the 14-dimensional adjoint multiplet which describes fields corresponding to physical particle states.

Employing the constrained Higgs multiplet approach to the spontaneously broken theory leads to a unique action (6) which contains the kinetic terms for the matter fields. We found that this action reduced to the promising flat space-time limit of a free field action for the massless multiplet of one spin $\frac{1}{2}$ and one spin 1 particle. However, unless the gauge fields vanish, the terms in the expansion (29), of our action in the unitary gauge, which diverge upon group contraction, are not topological invariants. The Higgs field approach to constructing matter Lagrangians appears therefore to be too restrictive to be of any value. The non-linear fields approach on the other hand was found to be too general to contribute to the basic problem of obtaining a supersymmetric action of the form (26) in the unitary gauge. In fact this approach serves to make very clear the limited impact that the OSp(1,4) gauge theory, broken to SL(2,c), can have on the construction of supersymmetric actions. The theory serves only to identify the vierbein fields and the $\frac{OSp(1,4)}{Sp(4)}$ sector appears to be of no consequence - it certainly does not correspond to the supersymmetry transformations which describe Fermi-Bose symmetries and are responsible for cancellations in quantum calculations. This supersymmetry must be imposed on the theory by a suitable <u>choice</u> of action in the unitary gauge.

APPENDIX A

WEYL, DIRAC AND MAJORANA SPINORS

The Lorentz group SO(1,3) is defined as the group of linear transformations which leave invariant the quadratic form,

$$\eta_{ab}\, x^a x^b \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.$$

The 4 × 4 Lorentz matrices Λ constitute the fundamental (vector) representation,

$$g \longrightarrow \Lambda(g) \qquad \text{for} \qquad g \in SO(1,3).$$

The action of SO(1,3) on x^a is then given by

$$x^a \xrightarrow{g} x'^a = \Lambda(g)^a{}_b\, x^b$$

and

$$x_a \xrightarrow{g} x'_a = \Lambda(g)_a{}^b\, x_b$$

where, by definition of SO(1,3), the $\Lambda^a{}_b$ and $\Lambda_a{}^b$ must satisfy $\underline{\Lambda_a{}^b \Lambda^a{}_c = \delta^b{}_c}$. These are 10 constraints on the 16 components of Λ so that sO(1,3) is a 6-dimensional group. The constraints may also be written,

$$\underline{\Lambda(g)^{-1}{}_a{}^b \equiv \Lambda(g^{-1})_a{}^b = \Lambda(g)^b{}_a} \,.$$

From the vector representation, higher dimensional tensor representations are defined in the usual manner[(33)]. These don't however include all the finite dimensional representations of the Lorentz group. More general representations are the spinor representations[(5)] which include the 'even' vector and tensor representations as a subset.

A Weyl 2-component spinor ϕ is a pair (ϕ_1, ϕ_2) of complex numbers, hence an element of $\mathbb{C}^2$, which is the fundamental representation space of $SL(2, \mathbb{C})$, (the group of all 2×2 complex matrices, M, with $\det M = 1$). A Weyl spinor ϕ therefore transforms under $g \in SL(2,\mathbb{C})$ as,

$$\underline{\phi \xrightarrow{g} \phi' = M(g)\phi} .$$

The Weyl spinors also give a representation of SO(1,3) through the group homomorphism $SL(2,c) \overset{2:1}{\cong} SO(1,3)$. This homomorphism is easily demonstrated using the 2×2 Hermitian Pauli matrices, σ^1, σ^2, σ^3 which, together with $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \sigma^o$ form a complete set $\{\sigma^a\}$ for the expansion of any 2×2 matrix. Now consider the Hermitian matrix, $X = x_a \sigma^a$ and notice that $\det X = x^a x_a$, so that any transformation which leaves $\det X$ invariant is, by definition, a Lorentz transformation. We may define the transformation of X under $SL(2,c)$ to be,

$$X \longrightarrow X' = M(g) X \, M(g)^{\dagger}$$

then X' is still Hermitian and may be written $X' = x'_a \sigma^a$, also $\det X' = \det X$ (since $\det M(g) = 1$), so that this $SL(2,c)$ transformation is also a Lorentz transformation,

$$x_a' \sigma^a = \Lambda_a{}^b x_b \sigma^a = M(g) x_b \sigma^b M(g)^{\dagger} .$$

We therefore identify the 2 : 1 homomorphism of SL(2,c) onto SO(1,3), through,

$$\underline{\sigma^a \Lambda_a{}^b = M(g) \sigma^b M(g)^{\dagger}} .$$

The fundamental representation $g \to M(g)$ of SL(2,c) also provides a representation of SO(1,3) (since $M = M(\Lambda)$, through the group homomorphism), we therefore call $g \in SO(1,3)$ and $g \to M(g)$, the

2-component Weyl spinor representation of SO(1,3). Introducing spinor indices $\alpha, \beta = 1, 2$ we write,

$$\phi_\alpha \xrightarrow{g} \phi_\alpha' = M(g)_\alpha{}^\beta \phi_\beta$$

where the indices are raised and lowered using the 2×2 symplectic metric[52], $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

This Weyl spinor representation is not self conjugate so that we also obtain the distinct <u>conjugate representation</u>,

$$\bar{\chi} \xrightarrow{g} \chi' = M(g)^* \chi \doteq \bar{\chi} M(g)^\dagger$$

and conventionally label the components with dotted indices $\dot{\alpha}, \dot{\beta} = 1,2$; $\chi_{\dot{\alpha}}' = M(g)^*{}_{\dot{\alpha}}{}^{\dot{\beta}} \chi_{\dot{\beta}}$.

Writing an element g of SO(1,3) or SL(2,c) as,

$$g = \exp(\tfrac{i}{2} \omega^{ab} M_{ab}), \quad \text{we identify } M_{ab} = - M_{ba}$$

as the group generators satisfying the usual (pseudo)-orthogonal group algebra,

$$[M_{ab}, M_{cd}] = i(\eta_{ac} M_{bd} + \eta_{bd} M_{ac} - \eta_{ad} M_{bc} - \eta_{bc} M_{ad}).$$

Then with $\bar{\sigma}^a \equiv \sigma_a = \eta_{ab}\sigma^b$ we may directly verify that

$$\pi^{ab} = -\tfrac{i}{4}(\sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a)$$

form a 2×2 spinorial matrix representation of the generators M_{ab}. Hence, under SO(1,3) the Weyl spinor ϕ_α transforms as, $\phi_\alpha \to \phi_\alpha' = \exp(\tfrac{i}{2} \omega^{ab} \pi_{ab})_\alpha{}^\beta \phi_\beta$. Similarly $\pi_{ab}^\dagger = -\tfrac{i}{4}(\bar{\sigma}^\beta \sigma^\alpha - \bar{\sigma}^\beta \sigma^\alpha)$ form the conjugate representation, $\chi_{\dot{\alpha}} \to \chi_{\dot{\alpha}}' \to \chi_{\dot{\beta}} \exp(-\tfrac{i}{2} \omega^{\alpha\beta} \pi_{\alpha\beta}^\dagger)^{\dot{\beta}}{}_{\dot{\alpha}}$.

The generators M_{ab} may be separated into rotations, $\underline{S} = -(M_{23}, M_{31}, M_{12})$ which form the SO(3) subgroup, and boosts, $\underline{T} = -(M_{01}, M_{02}, M_{03})$. Then defining

$$\underline{L} = \tfrac{1}{2}(\underline{S} + i\,\underline{T})$$

and

$$\underline{J} = \tfrac{1}{2}(\underline{S} - i\,\underline{T})$$

the Lorentz algebra takes the form,

$$[L_i, L_j] = i\,\varepsilon_{ijk}\,L_k$$

$$[L_i, J_j] = 0$$

$$[J_i, J_j] = i\,\varepsilon_{ijk}\,J_k.$$

Employing the standard SO(3) angular momentum theory the representations of SO(1,3) are labelled by two eigenvalues ℓ and $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. A general representation, (ℓ, j) has dimension $(2\ell + 1)(2j + 1)$ and has SO(3) spin content governed by the usual angular momentum coupling rules for $\underline{S} = \underline{L} + \underline{J}$, i.e. spin $S = |j - \ell|, \ldots |j + \ell|$. The two fundamental representations are, $(\frac{1}{2}, 0)$, to which ϕ_α belongs, and $(0, \frac{1}{2})$, to which $\chi_{\dot\alpha}$ belongs, and from these, higher dimensional representations are formed as tensor products. For example the 4-dimensional representation, $(\frac{1}{2}, \frac{1}{2})$ is described by $\phi_{\alpha\dot\beta}$ which may be written, using the Pauli matrices, as $\phi_{\alpha\dot\beta} = A_a \sigma^a_{\alpha\dot\beta}$. Hence the representation $(\frac{1}{2}, \frac{1}{2})$ is the 4-vector representation A_a $(a = 0,1,2,3)$ of SO(1,3) with SO(3) spin content, $S = 0, 1$.

<u>Dirac Spinors</u> ψ belong to the <u>direct sum</u> representation, $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of SO(1,3). Hence they are the four-component objects. $\psi = \binom{\phi}{\chi}$ and transform under SO(1,3) as

$$\psi \rightarrow \psi' = \begin{pmatrix}\phi'\\ \chi'\end{pmatrix} = \exp\left(\frac{i}{2}\,\omega_{ab}\begin{pmatrix}\pi_{ab} & 0\\ 0 & \pi_{ab}^{\dagger}\end{pmatrix}\right)\begin{pmatrix}\phi\\ \chi\end{pmatrix}$$

The 4×4 Dirac 'γ' matrices, required to satisfy

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}$$

may be defined (in the Weyl representation) by,

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix}, \qquad \text{then we may verify that}$$

$$\begin{pmatrix}\pi_{ab} & 0\\ 0 & \pi_{ab}^{\dagger}\end{pmatrix} = -\frac{i}{4}[\gamma_a, \gamma_b] \equiv -\tfrac{1}{2}\,\sigma_{ab}\,.$$ These matrices $-\frac{1}{2}\,\sigma_{ab}$ form the 4×4 Dirac spinor representation of the generators M_{ab}.

<u>Majorana Spinors</u> are self conjugate Dirac spinors satisfying $\psi = \alpha\psi^c$ where α is an arbitrary complex number and ψ^c is the charge conjugate spinor,

$$\psi^c = C\,\bar{\psi}^T \qquad (\bar{\psi} = \psi^{\dagger}\gamma_o \leftarrow \text{Dirac conjugate}).$$

The matrix C is the antisymmetric charge conjugation matrix and there exists a representation of the 'γ' matrices in which they are all pure imaginary and in which $C = k\gamma^o$ (where k is just a number). Hence in this 'Majorana representation' (see Appendix B) we see that, $\underline{\psi = \alpha'\,\psi^*}$, i.e. Majorana spinors are real, (with α' set equal to 1). More generally, we shall identify Majorana spinors as satisfying,

$$\underline{\bar{\psi} = \psi^{\dagger}\gamma_o = \psi^T C}\,,$$

throughout the main text.

APPENDIX B

THE DIRAC 'γ' MATRICES

The Dirac matrices γ_a are required, by definition, to satisfy

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab} \qquad (\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)).$$

In Appendix A we mentioned the Weyl representation,

$$\gamma_a = \left(\begin{array}{c|c} 0 & \sigma_a \\ \hline \sigma^a & 0 \end{array}\right)$$

where σ_i ($i = 1,2,3$) are the Pauli matrices.

There is also a representation of the Dirac matrices in which the γ_i are the same as those in the Weyl representation but γ_o is diagonal,

$$\gamma_o = \left(\begin{array}{c|c} \sigma_o & 0 \\ \hline 0 & -\sigma_o \end{array}\right), \qquad \sigma_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In the Majorana representation all the γ_a are pure imaginary,

$$\gamma_o = \left(\begin{array}{c|c} \sigma_2 & 0 \\ \hline 0 & -\sigma_2 \end{array}\right), \qquad \gamma_1 = i\left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & \sigma_1 \end{array}\right),$$

$$\gamma_2 = \left(\begin{array}{c|c} 0 & \sigma_2 \\ \hline -\sigma_2 & 0 \end{array}\right) \qquad \gamma_3 = i\left(\begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & \sigma_3 \end{array}\right).$$

More generally, all the irreducible representations of γ_a are connected by the similarity transformation,

$$\gamma_a' = S\,\gamma_a\,S^{-1},$$

given two representations γ_a, γ_a' then S is unique up to an arbitrary complex number. Now from the definition of the γ_a matrices it is clear that for any representation of the γ_a's then γ_a^T and $\gamma_a^\dagger$ will also give representations. Hence there must exist matrices C and K such that,

$$\gamma_a^T = -C\gamma_a C^{-1}$$

and

$$\gamma_a^\dagger = K\gamma_a K^{-1} .$$

We have chosen the minus sign in the first relation so that C may be identified as the charge conjugation matrix[(4)]. In the Majorana representation $C \sim \gamma_o$, up to an arbitrary complex number and in the main text we work with $C = \gamma_o = C^{-1}$.

In order that the Dirac Hamiltonian H ,$(H\psi = i\frac{\partial}{\partial t}\psi)$, be Hermitian it is necessary that $\gamma_o^\dagger = \gamma_o$ and $\gamma_i^\dagger = -\gamma_i$. Hence we shall always take $K = \gamma_o = K^{-1}$ (since $\gamma_o\gamma_i\gamma_o = -\gamma_i$), a choice which is stable under any unitary transformation, $\gamma_a \rightarrow U\gamma_a U^\dagger$.

From the γ_a we form a set,

$$\{\gamma_X\} = \{I, \gamma_a, \gamma_5, i\gamma_a\gamma_5, \sigma_{ab} = \tfrac{i}{2}[\gamma_a, \gamma_b]\}$$

of sixteen matrices (X = 1, ..., 16). These matrices are orthogonal in the sense that,

$$\frac{1}{4}\,\mathrm{Tr}\,\gamma_X\gamma^Y = (N_{XX})\delta_X^Y$$

where $N_{XX} = +1$ for $\gamma_X = I, \gamma_a, \sigma_{ab}$

$= -1$ for $\gamma_X = \gamma_5, i\gamma_a\gamma_5$

(note that $\gamma_5 \equiv \gamma^o\gamma^1\gamma^2\gamma^3$).

The γ_X are also linearly independent:

consider the linear relation, $$\sum_{X=1}^{16} a_X \gamma_X = 0$$

then this implies that $$\mathrm{Tr} \sum_{X=1}^{16} a_X \gamma_X \gamma^Y = 0$$

but $\mathrm{Tr}\, \Sigma\, a_X \gamma_X \gamma^Y = 4\, a^Y (N_{YY})$, for all $1 \leq Y \leq 16$. Hence the only solution to $\sum_{X=1}^{16} a_X \gamma_X$ is that all $a_X = 0$ which means that the γ_X are linearly independent.

It therefore follows that any 4×4 matrix M may be written as,

$$M = \sum_{X=1}^{16} a^X \gamma_X$$

then multiplying by γ^Y and taking the trace we see that,

$$a^X = \frac{1}{4N_{XX}} \mathrm{Tr}\, \gamma^X M$$

hence

$$M = \sum_X \frac{1}{4N_{XX}} \mathrm{Tr}(\gamma^X M) \gamma_X \, .$$

In component form this reads,

$$M_\alpha{}^\beta = \sum_X \frac{1}{4N_{XX}} (\gamma^X)_\delta{}^\varepsilon \, M_\varepsilon{}^\delta \, (\gamma_X)_\alpha{}^\beta$$

and as a special case,

$$\delta_\alpha{}^\varepsilon \, \delta_\kappa{}^\beta = \sum_X \frac{1}{4N_{XX}} (\gamma^X)_\kappa{}^\varepsilon (\gamma_X)_\alpha{}^\beta$$

which is the completeness relation for the matrices $\{\gamma_X\}$.

Finally, multiplying the completeness relation through by the spinors $\phi_\varepsilon \, \bar{\psi}^\kappa$ we find, (for Majorana spinors)

$$\phi_\alpha \, \bar{\psi}^\beta = \sum \frac{1}{4N_{XX}} (\Phi^T (C\gamma^X)^T \, \psi)(\gamma_X)_\alpha{}^\beta$$

or, writing out the γ^X explicitly,

$$\phi\bar{\psi} \quad = \quad \tfrac{1}{4}\{-\ \bar{\phi}\psi I + (\bar{\phi}\gamma_5\psi)\gamma_5 + \bar{\phi}i\gamma^a\gamma_5\psi\ i\gamma_a\gamma_5$$

$$+ (\bar{\phi}\gamma^a\psi)\gamma_a + \tfrac{1}{2}(\bar{\phi}\sigma^{ab}\psi)\sigma_{ab}\} \quad .$$

This is the basic Fierz resummation for the outer product of two Majorana spinors.

APPENDIX C

THE DENSITIES g, h, $\varepsilon^{\mu\nu\kappa\lambda}$ AND THE EINSTEIN ACTION

We define,

$$g \equiv \det g_{\mu\nu} < 0, \quad \text{so that} \quad g^{-1} = \det g^{\mu\nu}.$$ Then

with $g_{\mu\nu} = h_{\mu a}\, h_{\nu b}\, \eta^{ab}$ it follows that,

$$h \equiv \det h_{\mu a} = \pm \sqrt{-g} \tag{1}$$

and

$$\det h^{\mu}{}_{a} = \det g^{\mu\nu} h_{\nu a} = g^{-1} h = -h^{-1} \quad \text{etc.}$$

The variation of the determinant of any square non-singular matrix M is,

$$\delta(\det M) = (\det M)\,\mathrm{Tr}\, M^{-1}\,\delta M + O(\delta^2).$$

Hence,

$$\delta g = g\, g^{\mu\nu}\,\delta g_{\mu\nu} = -\, g\, g_{\mu\nu}\,\delta g^{\mu\nu} \tag{2a}$$

and

$$\delta h = h\, h^{\mu a}\,\delta h_{\mu a} = -\, h\, h_{\mu a}\,\delta h^{\mu a}\ . \tag{2b}$$

We shall define the totally antisymmetric tensor in 4-dimensional space-time so that $\varepsilon^{0123} = +1$ <u>in all bases</u> (coordinate and non-coordinate). Under coordinate transformations (between coordinate bases) $x^{\mu} \to x'^{\mu}$ then,

$$\varepsilon^{\mu\nu\kappa\lambda} \to \varepsilon'^{\mu\nu\kappa\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\rho}}\,\frac{\partial x'^{\nu}}{\partial x^{\sigma}}\,\frac{\partial x'^{\kappa}}{\partial x^{\tau}}\,\frac{\partial x'^{\lambda}}{\partial x^{\omega}}\,\varepsilon^{\rho\sigma\tau\omega}$$

so that

$$\varepsilon'^{0123} = \det\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right) \equiv \left|\frac{\partial x'}{\partial x}\right| \quad \text{(the Jacobian)}$$

and hence

$$\varepsilon^{\mu\nu\kappa\lambda} \doteq \left|\frac{\partial x}{\partial x'}\right| \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} \frac{\partial x'^{\kappa}}{\partial x^{\tau}} \frac{\partial x'^{\lambda}}{\partial x^{\omega}} \varepsilon^{\rho\sigma\tau\omega}$$

$\varepsilon^{\mu\nu\kappa\lambda}$ is therefore a contravariant tensor density of weight 1.

Now with $\varepsilon_{\mu\nu\kappa\lambda} \equiv g_{\mu\rho}\, g_{\nu\sigma}\, g_{\kappa\tau}\, g_{\lambda\omega}\, \varepsilon^{\rho\sigma\tau\omega}$

then $\varepsilon_{0123} = \det g_{\mu\nu} \equiv g,$

hence $\varepsilon_{\mu\nu\kappa\lambda} = g\, \varepsilon^{\mu\nu\kappa\lambda}\ .$ (3)

In particular for Minkowski space time, $\varepsilon_{abcd} = -\varepsilon^{abcd}$. More generally the relation between $\varepsilon^{\mu\nu\kappa\lambda}$ (coordinate basis) and ε^{abcd} (Lorentz basis) is determined as follows:

By definition, $h \equiv \det h_{\mu a} \equiv h_{oa}\, h_{1b}\, h_{2c}\, h_{3d}\, \varepsilon^{abcd}$

$$\therefore\ \varepsilon_{\mu\nu\kappa\lambda} = g\, \varepsilon^{\mu\nu\kappa\lambda} = \frac{g}{h}\, h_{\mu a}\, h_{\nu b}\, h_{\kappa c}\, h_{\lambda d}\, \varepsilon^{abcd}$$

i.e. $\varepsilon_{\mu\nu\kappa\lambda} = -h\, h_{\mu a}\, h_{\nu b}\, h_{\kappa c}\, h_{\lambda d}\, \varepsilon^{abcd}$, etc.

With this result we see that,

$$\varepsilon^{\mu\nu\kappa\lambda}\, \varepsilon_{\mu\nu\kappa\lambda} = h^2\, \varepsilon^{abcd}\, \varepsilon_{abcd} = 24g \tag{4}$$

which agrees with equation (3).

In general we see that,

$$\begin{aligned} \varepsilon^{\mu\nu\kappa\lambda}\, \varepsilon_{abcd} &= -\, h\, h^{\mu}{}_{\ell}\, h^{\nu}{}_{m}\, h^{\kappa}{}_{n}\, h^{\lambda}{}_{p}\, \varepsilon^{\ell mnp}\, \varepsilon_{abcd} \\ &= +\, h\, h^{\mu}{}_{\ell}\, h^{\nu}{}_{m}\, h^{\kappa}{}_{n}\, h^{\lambda}{}_{p}\, \varepsilon^{\ell mnp}_{[abcd]} \end{aligned}$$

$$\therefore\ \varepsilon^{\mu\nu\kappa\lambda}\, \varepsilon_{abcd} = h\, h^{\mu}{}_{[a}\, h^{\nu}{}_{b}\, h^{\kappa}{}_{c}\, h^{\lambda}{}_{d]} \tag{5a}$$

then

$$h_{\mu}{}^{a}\, \varepsilon^{\mu\nu\kappa\lambda}\, \varepsilon_{abcd} = h\, h^{\nu}{}_{[b}\, h^{\kappa}{}_{c}\, h^{\lambda}{}_{d]} \tag{5b}$$

and

$$h_\mu{}^a h_\nu{}^b \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{abcd} = 2h\, h^\kappa{}_{[c} h^\lambda{}_{d]} \tag{5c}$$

and

$$h_\mu{}^a h_\nu{}^b h_\kappa{}^c \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{abcd} = 6\, h\, h^\lambda{}_d \, . \tag{5d}$$

Any differential 4-form, Ω, on 4-dimensional space-time may be written,

$$\begin{aligned}\Omega(x) &= f(x)\, \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \\ &= \frac{1}{4!} f(x)\, \varepsilon^{ijk\ell}\, \theta^i \wedge \theta^j \wedge \theta^k \wedge \theta^\ell \\ &= \frac{1}{4!} f(x) g\, e_{ijk\ell}\, \theta^i \wedge \theta^j \wedge \theta^k \wedge \theta^\ell\end{aligned}$$

where $g = \det g_{ij}$ and $\{\theta^i\}$ is a general basis for 1-forms. Now on flat Minkowski space-time a volume integral over a region $\mathcal{U}$ takes the form,

$$\begin{aligned}V_{\mathcal{U}} &= \int_{\mathcal{U}} d^4x = \int_{\mathcal{U}} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ &= -\frac{1}{4!} \int_{\mathcal{U}} \varepsilon_{abcd}\, dx^a \wedge dx^b \wedge dx^c \wedge dx^d\end{aligned}$$

where x^a $(a = 0, 1, 2, 3)$ are Lorentz coordinates. On a curved manifold the volume integral has the same form but the Lorentz basis is no longer a coordinate basis so that $dx^a \to \theta^a$ and

$$\begin{aligned}V_{\mathcal{U}} &= -\frac{1}{4!} \int_{\mathcal{U}} \varepsilon_{abcd}\, \theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d \\ &= -\frac{1}{4!} \int_{\mathcal{U}} \varepsilon_{abcd}\, h_\mu{}^a h_\nu{}^b h_\kappa{}^c h_\lambda{}^d\, dx^\mu \wedge dx^\nu \wedge dx^\kappa \wedge dx^\lambda \\ V_{\mathcal{U}} &= -\int_{\mathcal{U}} (\det h_{\mu a})\, dx^4\end{aligned}$$

where we are now integrating over general coordinates x^μ. Hence we have just shown that the volume measure on curved (or flat) space-time is $\underline{dV = |h| d^4x = |\sqrt{-g}|\, d^4x}$, in terms of general coordinates x^μ.

The Einstein Action

The action for pure gravitation is,

$$I_E = \int d^4x \sqrt{-g}\; R = \int d^4x\, h\, h^\mu{}_a\, h^\nu{}_b\, R_{\mu\nu}{}^{ab}.$$

Using (5c) we see that I_E may also be written,

$$I_E = \int d^4x \frac{1}{4} \varepsilon^{\mu\nu\kappa\lambda}\, \varepsilon_{abcd}\, h_\mu{}^a\, h_\nu{}^b\, R_{\kappa\lambda}{}^{cd}.$$

We now calculate the variations, $\dfrac{\delta I_E}{\delta h_{\mu a}}$ and $\dfrac{\delta I_E}{\delta B_\mu{}^{ab}}$.

$$\delta_h \mathcal{L}_E = \delta(h\, h^\mu{}_a\, h^\nu{}_b) R_{\mu\nu}{}^{ab}$$

$$= \delta h^\mu{}_a(-h\, h_\mu{}^a\, R + 2h\, h^\nu{}_b\, R_{\mu\nu}{}^{ab})$$

i.e. $$\underline{\frac{\delta I_E}{\delta h_{\mu a}} = 2h(R_{\mu a} - \frac{1}{2} h_{\mu a} R)} . \qquad (6)$$

$$\delta_B \mathcal{L}_E = h\, h^\mu{}_a\, h^\nu{}_b\, \delta(R_{\mu\nu}{}^{ab})$$

$$= h\, h^\mu{}_{[a}\, h^\nu{}_{b]}\, (\partial_\mu\, \delta B_\nu{}^{ab} + (\delta\, B_\mu{}^{ac}) B_\nu{}^b{}_c + B_\mu{}^{ac}\, \delta B_\nu{}^b{}_c)$$

but

$$D_\mu(\delta B_\nu{}^{ab}) = \partial_\mu(\delta B_\nu{}^{ab}) - B_\mu{}^a{}_c\, \delta B_\nu{}^{cb} - B_\mu{}^b{}_c\, \delta B_\nu{}^{ac}$$

$$\therefore \quad \delta_B \mathcal{L}_E = h\, h^\mu{}_{[a}\, h^\nu{}_{b]}\, D_\mu(\delta B_\nu{}^{ab}).$$

Now for any group invariant $I(U,V) \equiv U.V$ where U and V are co-variants then $\delta I = 0 \longrightarrow \delta U \cdot V = -U \cdot \delta V$.

Hence
$$D_\mu I \equiv \partial_\mu I = (\partial_\mu U)\cdot V + U \cdot \partial_\mu V$$
$$= (D_\mu U)\cdot V + U \cdot D_\mu V ,$$

and we may quite generally integrate by parts over covariant derivatives,

$$U\cdot D_\mu V \quad = \quad \text{total divergence} - (D_\mu U)\cdot V.$$

Applying this result to $\delta_B \mathcal{L}_E$ (a local Lorentz invariant) we see that,

$$\delta_B \mathcal{L}_E \quad = \quad \text{total divergence} - D_\mu (h\, h^\mu{}_{[a} h^\nu{}_{b]})\delta B_\nu{}^{ab}$$

then using (5c) and dropping the total divergence,

$$\underline{\delta_B \mathcal{L}_E \quad = \quad -\varepsilon^{\mu\nu\kappa\lambda}\, \varepsilon_{abcd}(D_\mu h_\kappa{}^c)h_\lambda{}^d\ \delta B_\nu{}^{ab}} \tag{7}$$

and now using (5b)

$$= \ - h\, h^\mu{}_{[a}\, h^\nu{}_b\, h^\kappa{}_{c]}\ (D_\mu h_\kappa{}^c)\delta B_\nu{}^{ab}$$

$$= \ - h(S_{a\kappa}{}^\kappa\, h^\nu{}_b + S_{\mu b}{}^\mu\, h^\nu{}_a + S_{ba}{}^\nu)\ \delta B_\nu{}^{ab}$$

where $S_{\mu\nu}{}^a \ = \ D_\mu h_\nu{}^a - D_\nu h_\mu{}^a$ is the torsion tensor field and
$S_{a\nu}{}^b \ = \ h^\mu{}_a\, S_{\mu\nu}{}^b$, etc.

We have therefore found that,

$$\underline{\frac{\delta I_E}{\delta B_\nu{}^{ab}} \quad = \quad h(S_{ab}{}^\nu - h^\nu{}_a\, S_{\mu b}{}^\mu + h^\nu{}_b\, S_{\mu a}{}^\mu)} \tag{8}$$

<u>The Supergravity Field Equations</u>

We illustrate the case of gravity coupled to matter using the supergravity Lagrangian,

$$\underline{\mathcal{L}_{SG} \ = \ \frac{1}{\kappa^2}\, h\, h^\mu{}_a\, h^\nu{}_b\, R_{\mu\nu}{}^{ab} - 2i\ \varepsilon^{\mu\nu\kappa\lambda}\ \bar{\psi}_\mu \gamma_5 \gamma_\nu D_\kappa \psi_\lambda\ .} \tag{9}$$

Using (6) we easily find that,

$$\frac{\delta I_{SG}}{\delta h_{\mu a}} = \frac{2h}{\kappa^2}(R_{\mu a} - \frac{1}{2} h_{\mu a} R) + 2i\, \varepsilon_{\mu\nu\kappa\lambda} \bar{\psi}^{\nu} \gamma_5 \gamma_a D^{\kappa} \psi^{\lambda}$$

so that the field equations for the vierbein fields are

$$\underline{R_{\mu a} - \frac{1}{2} h_{\mu a} R = \frac{-i\kappa^2}{h} \varepsilon_{\mu\nu\kappa\lambda} \bar{\psi}^{\nu} \gamma_5 \gamma_a D^{\kappa} \psi^{\lambda}} \quad . \tag{10}$$

For the B^{ab} we see that,

$$\delta_B \mathcal{L}_{SG} = \frac{1}{\kappa^2} h\, h^{\mu}{}_a\, h^{\nu}{}_b\, \delta R_{\mu\nu}{}^{ab} - 2i\, \varepsilon^{\mu\nu\kappa\lambda} \bar{\psi}_{\mu} \gamma_5 \gamma_{\nu} \frac{i}{4} \delta(B_{\kappa}{}^{ab}) \sigma_{ab} \psi_{\lambda}$$

then using (7),

$$= \varepsilon^{\mu\nu\kappa\lambda} (\frac{1}{2\kappa^2} \varepsilon_{abcd}\, S_{\nu\kappa}{}^{c}\, h_{\lambda}{}^{d} - \frac{1}{2} \bar{\psi}_{\kappa} \gamma_5 \gamma_{\nu} \sigma_{ab} \psi_{\lambda}) \delta B_{\mu}{}^{ab} .$$

Now $\varepsilon^{\mu\nu\kappa\lambda} \bar{\psi}_{\kappa} \gamma_5 \gamma_{\nu} \sigma_{ab} \psi_{\lambda} = -i\, \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{abcd}\, h_{\nu}{}^{c}\, \bar{\psi}_{\kappa} \gamma^{d} \psi_{\lambda}$

$$\therefore \quad \delta_B \mathcal{L}_{SG} = \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{abcd}\, h_{\lambda}{}^{d} (\frac{1}{2\kappa^2} S_{\nu\kappa}{}^{c} + \frac{i}{2} \bar{\psi}_{\kappa} \gamma^{c} \psi_{\nu}) \delta B_{\mu}{}^{ab}$$

then using (5b)

$$= h\, h^{\mu}{}_{[a}\, h^{\nu}{}_{b}\, h^{\kappa}{}_{c]} (\frac{1}{2\kappa^2} S_{\nu\kappa}{}^{c} - \frac{i}{2} \bar{\psi}_{\nu} \gamma^{c} \psi_{\kappa}) \delta B_{\mu}{}^{ab} \quad .$$

Hence the field equations for the $B_{\mu}{}^{ab}$ are,

$$\underline{S_{\mu\nu}{}^{a} \equiv D_{\mu} h_{\nu}{}^{a} - D_{\nu} h_{\mu}{}^{a} = i\kappa^2\, \bar{\psi}_{\mu} \gamma^{a} \psi_{\nu}} \quad . \tag{11}$$

and give the torsion $S_{\mu\nu}{}^{a}$ algebraically in terms of the spin $\frac{3}{2}$ fields ψ_{μ}. Equations (11) may be solved for $B_{\mu}{}^{ab}$ by writing, $B_{\mu}{}^{ab} = B_{\mu}{}^{ab}(h) + K_{\mu}{}^{ab}$ where $B_{\mu}{}^{ab}(h)$ is the zero-torsion connection (I.44), i.e. the solution of $D_{\mu} h_{\nu}{}^{a} - D_{\nu} h_{\mu}{}^{a} = 0$. With this substitution (11) becomes

$$K_{\mu}{}^{a}{}_{b}\, h_{\nu}{}^{b} - K_{\nu}{}^{a}{}_{b}\, h_{\mu}{}^{b} \equiv K_{\mu}{}^{a}{}_{\nu} - K_{\nu}{}^{a}{}_{\mu} = -i\kappa^2\, \bar{\psi}_{\mu} \gamma^{a} \psi_{\nu} \quad .$$

Rewriting this equation twice with a cyclic permutation of indices then subtracting the third from the sum of the first two gives,

$$2\,K_{\mu\ \nu}^{\ a} \quad = \quad \kappa^2(\bar{\psi}_\mu \gamma^a \psi_\nu + \bar{\psi}^a \gamma_\nu \psi_\mu - \bar{\psi}_\nu \gamma_\mu \psi^a)$$

hence,

$$\underline{B_\mu^{\ ab}(h,\psi) \quad = \quad B_\mu^{\ ab}(h) - \tfrac{i}{2}\kappa^2\ (\bar{\psi}_\mu \gamma^a \psi^b + \bar{\psi}^a \gamma^b \psi_\mu + \bar{\psi}^a \gamma_\mu \psi^b)}\ . \qquad (12)$$

Finally, for the ψ_μ field we see that

$$\delta_\psi \mathcal{L}_{SG} \quad = \quad -2i\ \varepsilon^{\mu\nu\kappa\lambda}(\delta\bar{\psi}_\mu\ \gamma_5\gamma_\nu D_\kappa \psi_\lambda + \bar{\psi}_\mu \gamma_5 \gamma_\nu D_\kappa \delta\psi_\lambda).$$

The second term $=$ total derivative $+\ 2i\ \varepsilon^{\mu\nu\kappa\lambda}(\delta\bar{\psi}_\lambda \gamma_5 (D_\kappa h_\nu^{\ a})\gamma_a \psi_\mu$

$$+ \quad \delta\bar{\psi}_\lambda\ \gamma_5\gamma_\nu\ D_\kappa \psi_\mu),$$

then using (11) and the identity, $\varepsilon^{\mu\nu\kappa\lambda}\ \bar{\psi}_\nu \gamma^a \psi_\kappa\ \gamma_a \psi_\mu \ \equiv\ 0$ (which is verified using the Fierz resummation), we find that,

$$\delta_\psi \mathcal{L}_{SG} \quad = \quad -2i\ \varepsilon^{\mu\nu\kappa\lambda}(\delta\bar{\psi}_\mu \gamma_5 \gamma_\nu D_\kappa \psi_\lambda - \delta\bar{\psi}_\lambda \gamma_5 \gamma_\nu D_\kappa \psi_\mu)\ .$$

Hence the field equations for ψ_μ are,

$$\underline{R^\mu \quad \equiv \quad \varepsilon^{\mu\nu\kappa\lambda}\ \gamma_5 \gamma_\nu \psi_{\kappa\lambda} \quad = \quad 0} \qquad (13)$$

where $\underline{\psi_{\kappa\lambda} \ = \ D_\kappa \psi_\lambda - D_\lambda \psi_\kappa}$.

APPENDIX D

NON-LINEAR REALIZATIONS AND THE EINSTEIN ACTION

a) Gravity as a Spontaneously Broken Gauge Theory of SO(2,3)

SO(2,3) has 10 generators, $M_{AB} = - M_{BA}$ (A,B = 0,1,2,3,5), satisfying the Lie algebra.

$$[M_{AB}, M_{CD}] = i(\eta_{AC} M_{BD} + \eta_{BD} M_{AC} - \eta_{AD} M_{BC} - \eta_{BC} M_{AD})$$

where $\eta_{AB} = \mathrm{diag}(+1, -1, -1, -1, +1)$.

The M_{ab} (a,b = 0,1,2,3) generate the Lorentz subgroup SO(1,3) and we decompose the above relation into,

$$[M_{ab}, M_{cd}] = i(\eta_{ac} M_{bd} + \eta_{bd} M_{ac} - \eta_{ad} M_{bc} - \eta_{bc} M_{ad}) \tag{1a}$$

$$[M_{ab}, M_{5d}] = i(\eta_{ad} M_{5b} - \eta_{bd} M_{5a}) \tag{1b}$$

$$[M_{5b}, M_{5d}] = i\, M_{bd}. \tag{1c}$$

The SO(2,3) gauge potentials are the components of the connection form, $\omega = \omega_\mu dx^\mu$ where

$$\omega_\mu = \frac{i}{2} B_\mu{}^{AB} M_{AB} \equiv \frac{i}{2} B_\mu{}^{ab} M_{ab} + i\, B_\mu{}^{5a} M_{5a} \quad . \tag{2}$$

Under $g \in SO(2,3)$, ω_μ transforms as,

$$\omega_\mu \xrightarrow{g} \omega_\mu{}' = g\, \omega_\mu\, g^{-1} - g \partial_\mu\, g^{-1} \quad .$$

For infinitestimal transformations, $g = 1 + \frac{i}{2} \omega^{AB} M_{AB}$ (neglect $O(\omega^2)$)

$$\delta B_\mu{}^{AB} = \partial_\mu \omega^{AB} - (\omega^{AC} B_\mu{}^B{}_C - B_\mu{}^{AC} \omega^B{}_C)$$

i.e. $$\delta B_\mu{}^{ab} = \partial_\mu \omega^{ab} - (\omega^{ac} B_\mu{}^b{}_c - \omega^{bc} B_\mu{}^a{}_c) \qquad (3a)$$

and

$$\delta B_\mu{}^{5a} = \partial_\mu \omega^{5a} + \omega^a{}_c B_\mu{}^{5c} - \omega^{5c} B_\mu{}^a{}_c , \qquad (3b)$$

The $SO(2,3)$ covariant derivative ∇_μ is given by,

$$\nabla_\mu = \partial_\mu - \omega_\mu = D_\mu - i B_\mu{}^{5a} M_{5a}$$

where $D_\mu = \partial_\mu - \frac{i}{2} B^{ab} M_{ab}$ is the Lorentz covariant derivative. The field strengths, $B_{\mu\nu}{}^{AB}$ are then defined by,

$$[\nabla_\mu, \nabla_\nu] = \frac{i}{2} B_{\mu\nu}{}^{AB} M_{AB} = -(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu - [\omega_\mu, \omega_\nu])$$

so that

$$B_{\mu\nu}{}^{AB} = \partial_\mu B_\nu{}^{AB} - \partial_\nu B_\mu{}^{AB} + (B_\mu{}^{AC} B_\nu{}^B{}_C - B_\nu{}^{AC} B_\mu{}^B{}_C)$$

i.e. $$B_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} + (B_\mu{}^{5a} B_\nu{}^{5b} - B_\nu{}^{5a} B_\mu{}^{5b}) \qquad (4a)$$

and

$$B_{\mu\nu}{}^{5a} = D_\mu B_\nu{}^{5a} - D_\nu B_\mu{}^{5a} . \qquad (4b)$$

where $R_{\mu\nu}{}^{ab} = \partial_\mu B_\nu{}^{ab} - \partial_\nu B_\mu{}^{ab} + (B_\mu{}^{ac} B_\nu{}^b{}_c - B_\nu{}^{ac} B_\mu{}^b{}_c)$ are the $SO(1,3)$ field strengths, eqn. (1.36).

We now briefly review the model of West and Stelle[(64),(77)] for a gauge theory of $SO(2,3)$, spontaneously broken to $SO(1,3)$. The spontaneous symmetry breaking in this model occurs through the introduction of the auxiliary Higgs fields, $y_A(x)$ which form an $SO(2,3)$ 5-vector, subject to the $SO(2,3)$ invariant constraint,

$$y^A(x)\, y_A(x) = \eta_{AB}\, y^A(x)\, y^B(x) = R^2 = m^{-2} . \qquad (5)$$

This constraint prevents the components from all acquiring vacuum expectation values and so triggers the spontaneous symmetry breaking of SO(2,3) down to the stability subgroup, SO(1,3) of any point on the hypersphere $y^A y_A = R^2$.

An SO(2,3) invariant gauge action incorporating the Higgs multiplet $y_A(x)$ is,[69]

$$\mathcal{L} = \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{ABCDE} B_{\mu\nu}{}^{AB} B_{\kappa\lambda}{}^{CD} m y^E + \lambda(x)(y^A y_A - R^2) . \tag{6}$$

The $y_A(x)$ and $\lambda(x)$ are all auxiliarly fields which may be eliminated through their algebraic field equations to obtain an action non-polynomial in the $B_{\mu\nu}{}^{AB}$. Notice that the field equation for the scalar $\lambda(x)$ is simply the constraint (5).

Since we have a local gauge invariance and since the $y_A(x)$ form an SO(2,3) 5-vector, it follows that we may find a gauge in which $y_A = (0,0,0,0,R)$. In this gauge (unitary gauge) we see that the Lagrangian (6) becomes,

$$\mathcal{L}\Big]_{\text{unitary}} = \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{abcd} B_{\mu\nu}{}^{ab} B_{\kappa\lambda}{}^{cd} .$$

Substituting for $B_{\mu\nu}{}^{ab}$ from (4a)

$$\mathcal{L}\Big]_{u} = \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{abcd} \Big[R_{\mu\nu}{}^{ab} R_{\kappa\lambda}{}^{cd} + 4B_\mu{}^{5a} B_\nu{}^{5b} R_{\kappa\lambda}{}^{cd} + 4B_\mu{}^{5a} B_\nu{}^{5b} B_\kappa{}^{5c} B_\lambda{}^{5d} \Big] . \tag{7}$$

This may be identified as theory of gravitation if we call $h_\mu{}^a = m^{-1} B_\mu{}^{5a}$ the vierbein field, with $\left[h_\mu{}^a\right] = 0$ as required. The use of m as opposed to κ (gravitational constant) for rescaling $B_\mu{}^{5a}$ is fully justified in Chapter III by the requirement of the contraction of the adjoint representation of SO(2,3) as this group contracts $(m \to 0)$ to iso(1,3)).

The first term in (7) is the Gauss-Bonnet topological invariant:

$$\delta(\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,R_{\mu\nu}{}^{ab}\,R_{\kappa\lambda}{}^{cd}) = 2\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}(\delta R_{\mu\nu}{}^{ab})R_{\kappa\lambda}{}^{cd}$$

but in Appendix C we show that $\varepsilon^{\mu\nu\kappa\lambda}\,\delta\,R_{\mu\nu}{}^{ab} = \varepsilon^{\mu\nu\kappa\lambda}\,D_\mu(\delta\,B_\nu{}^{ab})$, therefore,

$$\delta(\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,R_{\mu\nu}{}^{ab}\,R_{\kappa\lambda}{}^{cd}) = 2\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,D_\mu(\delta\,B_\nu{}^{ab})R_{\kappa\lambda}{}^{cd}$$

$$= \text{total derivative} - 2\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,\delta\,B_\nu{}^{ab}\,D_\mu\,R_{\kappa\lambda}{}^{cd}\,.$$

Now $\varepsilon^{\mu\nu\kappa\lambda}\,D_\mu\,R_{\kappa\lambda}{}^{cd} = 0$ are the Bianchi identities for the Lorentz field strengths (see eqn. (3.11)) so that,

$$\delta(\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,R_{\mu\nu}{}^{ab}\,R_{\kappa\lambda}{}^{cd}) = \text{total derivative,}$$

for any variation of the spin connection potentials $B_\mu{}^{ab}$.

Dropping the Gauss-Bonnet topological invariant we see that,

$$\mathcal{L}\rfloor_\mu = 4m^2\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\left[h_\mu{}^a h_\nu{}^b R_{\kappa\lambda}{}^{cd} + m^2\,h_\mu{}^a h_\nu{}^b h_\kappa{}^c h_\lambda{}^d\right]\,.$$

The first term is the Einstein Lagrangian (Appendix C) so that we multiply $\mathcal{L}\rfloor_u$ through by a dimensionless coupling parameter $\frac{1}{\Lambda^2}$ and identity $\frac{\Lambda}{m}$ as the gravitational coupling constant κ.

$$\frac{1}{\Lambda^2}\mathcal{L}\rfloor_u = \frac{4}{\kappa^2}\,\varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,h_\mu{}^a\,h_\nu{}^b\,R_{\kappa\lambda}{}^{cd} + \frac{4m}{\kappa^2}\,(24\,\det h_\mu{}^a)$$

$$\frac{1}{\Lambda^2}\mathcal{L}\rfloor_u = \frac{16}{\kappa^2}\,(\sqrt{-g}\,R + 6m^2\,\sqrt{-g}\,) \qquad \text{(see Appendix C).}$$

Hence the SO(2,3) action, (6) reduces to the Einstein action with a cosmological constant, $-6m^2$. This cosmological term may be removed at this stage by letting $m \to 0$ (contraction of SO(2,3) to iso(1,3), see Chapter III).

b) Spontaneous Symmetry Breaking and the Non-Linear Realization of Groups on Their Coset Spaces

A Lie group G may be realized as the Lie transformation group acting on an n-dimensional manifold M_n. If $\{x^i\}$ $(i = 1, \ldots, n)$ are coordinates on M_n then the action of $g \in G$ on M_n may be written as

$$x^i \xrightarrow{g} x'^i = Tg\, x^i \equiv x'^i(g,x^j). \tag{8}$$

For fixed g we see that the non-linear operator Tg effects active general coordinate transformations where x^i and x'^i label different points in the same coordinate system. One special case of interest is when Tg is simply a linear matrix operator

$$x'^i = Tg\, x^i = T(g)^i{}_j\, x^j$$

and $g \longrightarrow T(g)^i{}_j$ is the $n \times n$ matrix representation of G. Any realization $g \longrightarrow T_g$ of G is required by definition to satisfy,

$$\underline{Tg_1\, Tg_2 = Tg_1g_2} \,. \tag{9}$$

Now consider a subgroup H of G and recall that we may partition G into left or right cosets, gH and Hg (for all $g \in G$). By selecting just one element from each coset in the partition we factor H out of G and obtain the coset space, $\frac{G}{H}$. It is therefore clear that any $g \in G$ may be written as,

$$\underline{g = ch} \quad \text{(for left cosets)} \tag{10a}$$

where $\quad h \in H \quad$ and $\quad c \in \frac{G}{H}$.

Let $\mathscr{g} \in$ Lie algebra of G with basis $\{X_I\}$ $(I = 1, \ldots, n)$, then we write $\mathscr{g}$ as,

$$\mathscr{g} = \xi^I X_I \equiv \xi^a T_a + \xi^i S_i = \mathscr{c} \oplus \mathscr{h}$$

where $\quad \mathscr{c} = \xi^a T_a \quad (a = 1, \ldots, n-m)$ and T_a are coset generators

$\mathscr{h} = \xi^i S_i \quad (i = 1, \ldots, m)$ and S_i are subgroup H generators.

The n-m parameters ξ^a are coordinates for the coset space, $\frac{G}{H}$, labelling each point $\underline{c = e^{\xi^a T_a}}$, so that (10a) becomes,

$$\underline{g = e^{\xi^a T_a} h} \tag{10b}$$

The non-linear realization of G on the $(n-m)$-dimensional manifold $\frac{G}{H}$ is then simply defined by the group multiplication, $g_1 g_2 = g_3$ which for (10b) reads,

$$g \, e^{\xi^a T_a} h = e^{\xi'^a T_a} h'$$

i.e.
$$\underline{g e^{\xi^a T_a} = e^{\xi'^a T_a} h_1} \qquad (h_1 = h' h^{-1}) . \tag{11}$$

This equation may be solved explicitly provided the algebra has the following (weakly reducible[48]) structure,

$$\underline{[\mathscr{h}, \mathscr{h}] \subset \mathscr{h}} \tag{12a}$$

$$\underline{[\mathscr{h}, \mathscr{c}] \subset \mathscr{c}} \tag{12b}$$

$$\underline{[\mathscr{c}, \mathscr{c}] \subset \mathscr{h}} \tag{12c}$$

.

The explicit solution of (11) will take the form,

$$\underline{\xi'^a = \xi'^a(\xi^a, g)} \tag{13a}$$

and

$$\underline{h_1 = h_1(\xi^a, g)} \tag{13b}$$

Clearly (13a) defines a realization of G on $\frac{G}{H}$ acting explicitly through the coordinates, ξ^a of $\frac{G}{H}$. (The definition (9) is satisfied automatically since this realization is defined by the group multiplication). In general the relations (13) will be non-linear, however for $g = h \varepsilon H$ we see that (11) becomes

$$h\, e^{\xi^a T_a} = e^{\xi'^a T_a}\, h_1$$

and the L.H.S. $= h\, e^{\xi^a T_a}\, h^{-1}\, h$

but (12b) implies that $h\, e^{\xi^a T_a}\, h^{-1} \varepsilon \frac{G}{H}$.

Hence, identifying the L.H.S. with $e^{\xi'^a T_a}\, h_1$ we see that,

$$e^{\xi'^a T_a} = h\, e^{\xi^a T_a}\, h^{-1} \quad \text{and} \quad h = h_1.$$

For $g = h \varepsilon H$ we therefore see that equations (13) become the linear relations,

$$\xi'^a T_a = \xi'^a(\xi^a, h) T_a = h\, \xi^a T_a\, h^{-1} = (T(h)^a{}_b\, \xi^b).T_a$$

and

$$h_1 = h_1(\xi^a, h) = h.$$

Non-Linear Fields[13]

Suppose we have fields (coordinates) Φ_α, which transform as,

$$\Phi_\alpha \xrightarrow{g} \Phi'_\alpha = T_g\,\Phi_\alpha \tag{14}$$

then we define new fields, $\hat{\Phi}_\alpha = \hat{\Phi}_\alpha(\Phi_\alpha, \xi)$ by

$$\hat{\Phi}_\alpha \equiv T_{e^{-\xi^a T_a}}\,\Phi_\alpha \quad . \tag{15}$$

The transformation of the $\hat{\Phi}_\alpha$ under $g \in G$ is then,

$$\hat{\Phi}_\alpha \xrightarrow{g} \hat{\Phi}'_\alpha = T_{e^{-\xi'^a T_a}}\,\Phi'_\alpha$$

$$= T_{e^{-\xi'^a T_a}}\,T_g\,\Phi_\alpha$$

$$= T_{h_1}\,T_{e^{-\xi^a T_a}}\,\Phi_\alpha \quad \text{(using (11) and (9))}$$

i.e. $$\hat{\Phi}_\alpha' = T_{h_1}\,\hat{\Phi}_\alpha \qquad (h_1 = h_1(g,\xi_a)) \quad . \tag{16}$$

The fields $\hat{\Phi}_\alpha(\Phi_\alpha, \xi^a)$ thus transform under the full group, G, only by elements $h_1(g, \xi^a)$ of the subgroup H. The significance of these fields is well illustrated in the following two examples.

Example (a) If Φ_α form a vector space for the linear representation $\Phi_\alpha' = T(g)_\alpha^{\ \beta}\,\Phi_\beta$ then the non-linear fields $\hat{\Phi}_\alpha$ are defined by

$$\hat{\Phi}_\alpha = T(e^{-\xi^a T_a})_\alpha^{\ \beta}\,\Phi_\beta$$

and transform as,

$$\underline{\hat{\Phi}_\alpha \longrightarrow \hat{\Phi}'_\alpha = T(h_1(g, \xi^a))_\alpha{}^\beta \hat{\Phi}_\beta .} \tag{17a}$$

If $T(g)$ is an irreducible matrix representation of G then $T(h_1)$ will be a reducible representation of H and may be transformed (by a similarity transformation) into the block diagonal form,

$$\begin{pmatrix} \hat{\Phi}^{(1)} \\ \hat{\Phi}^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}' = \begin{pmatrix} T(h_1)^{(1)} & 0 & 0 \; . \; . \; . \; . \\ 0 & T(h_1)^{(2)} & 0 \\ & 0 & \\ 0 & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \end{pmatrix} \begin{pmatrix} \hat{\Phi}^{(1)} \\ \hat{\Phi}^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \tag{17b}$$

The $\hat{\Phi}^{(i)}$ form irreducible representations of H and don't mix with each other even under the full group G. (The components of Φ mix under a transformation $g \in G$.)

<u>Example (b)</u> Now consider the components, $\omega_\mu{}^I$ of the connection form $\omega = \omega_\mu{}^I X_I dx^\mu$ which transform under $g \in G$ according to,

$$\omega_\mu \xrightarrow{g} \omega_\mu' = Tg\, \omega_\mu = g\, \omega_\mu g^{-1} - g\, \partial_\mu g^{-1}.$$

We therefore define the non-linear gauge potentials as the components of,

$$\underline{\hat{\omega}_\mu = T_{e^{-\xi^a T_a}}\, \omega_\mu = e^{-\xi^a T_a} \omega_\mu e^{\xi^a T_a} - e^{-\xi^a T_a} \partial_\mu e^{\xi^a T_a}} \tag{18a}$$

and we may verify directly from (11) that $\hat{\omega}_\mu$ transforms as,

$$\underline{\hat{\omega}_\mu \xrightarrow{g} \omega_\mu' = h_1 \omega_\mu h_1^{-1} - h_1 \partial_\mu h_1^{-1} .} \tag{18b}$$

Now,

$$\hat{\omega}_\mu = \hat{\omega}_\mu{}^a T_a + \hat{\omega}_\mu{}^i S_i$$

so that, using (12b) we may immediately decompose (18b) into

$$\underline{\omega_\mu{}'^a T_a = h_1 \omega_\mu{}^a T_a h_1{}^{-1}} \qquad (19a)$$

and

$$\underline{\omega_\mu{}'^i S_i = h_1 \omega_\mu{}^i S_i h_1{}^{-1} - h_1 \partial_\mu h_1{}^{-1}} \qquad (19b)$$

and we observe that the non-linear gauge potentials $\hat{\omega}_\mu{}^a$, corresponding to the coset generators have homogeneous transformations.

Spontaneous Symmetry Breaking

Here we are interested in field theories in which the action is invariant under G but the ground state of the system is only invariant under a subgroup H. Physically equivalent 'vacuum states' are connected via the action of the $C \in \frac{G}{H}$ and, in fact, the orbit of the vacuum states (under the action of G) is in one-one correspondence with the coset space $\frac{G}{H}$. It follows therefore that within the framework of the non-linear realization of symmetry groups on their coset spaces, the coordinates, ξ_a, of the coset space play the role of the Goldstone modes in a field theory[64]. If the G-invariance is a local symmetry then the coordinates ξ^a are a set of fields $\xi^a = \xi^a(x^\mu)$ which may be gauged away by the transformation $g = e^{-\xi^a(x).T_a}$ (see equation (11)). The gauge in which these Goldstone fields vanish is known as the unitary gauge and we see from (15) that in this gauge the non-linear fields $\hat{\Phi}$ coincide with Φ.

Non-Linear Lagrangians

The transformation properties (16) of the non-linear fields, $\hat{\Phi}_\alpha(\Phi_\alpha, \xi^a)$ enable us to construct actions from these fields which need only be linearly invariant under the subgroup H to be invariant under the full non-linear transformations of G, with $h \to h_1(g, \xi^a)$. These actions implicitly contain the Goldstone fields $\xi^a(x)$ through the definition (15) of the non-linear fields. Gauging away the Goldstone fields leaves an action which is only invariant under the subgroup H.

(c) Non-Linear Actions for Gravity

The problem of constructing a gauge theory of SO(2,3) spontaneously broken to SO(1,3) is now approached using the general theory of section (b). We are interested in the non-linear realization of SO(2,3) on $\frac{SO(2,3)}{SO(1,3)}$. The relevant decomposition of the algebra is,

$$\mathfrak{g} = \frac{i}{2}\ {}^{AB} M_{AB} = i\, \omega^{5a} M_{5a} + \frac{i}{2} \omega^{ab} M_{ab} = \mathfrak{b} \oplus \mathfrak{h} .$$

An element g of SO(2,3) is thus written as

$$g = e^{i\omega^{5a} M_{5a}} h \qquad (h \in SO(1,3))$$

and the $\omega^{5a}(x)$ are the four Goldstone fields in the system. The non-linear gauge potentials for SO(2,3) are,

$$\hat{\omega}_\mu = i\, \hat{B}_\mu{}^{5a} M_{5a} + \frac{i}{2} \hat{B}_\mu{}^{ab} M_{ab}$$

and the non-linear field strengths are defined by

$$\left[\hat{\nabla}_\mu, \hat{\nabla}_\nu\right] = -\frac{i}{2} \hat{B}_{\mu\nu}{}^{AB} M_{AB} = -(\partial_\mu \hat{\omega}_\nu - \partial_\nu \hat{\omega}_\mu - \left[\hat{\omega}_\mu, \hat{\omega}_\nu\right])$$

so that,

$$\hat{B}_{\mu\nu}{}^{ab} = \hat{R}_{\mu\nu}{}^{ab} + (\hat{B}_{\mu}{}^{5a}\,\hat{B}_{\nu}{}^{5b} - \hat{B}_{\nu}{}^{5a}\,\hat{B}_{\mu}{}^{5b})$$

and

$$\hat{B}_{\mu\nu}{}^{5a} = \hat{D}_{\mu}\,\hat{B}_{\nu}{}^{5a} - \hat{D}_{\nu}\,\hat{B}_{\mu}{}^{5a} .$$

(where $\hat{D}_{\mu} \equiv \partial_{\mu} - \frac{i}{2}\,\hat{B}_{\mu}{}^{ab}\,M_{ab}$)

The geometrical significance of this theory is discussed in Chapter I, here we are only interested in writing down non-linear actions. These actions are identified as $SO(1,3)$ invariants formed from the covariant field strengths $\hat{B}_{\mu\nu}{}^{ab}$ and $\hat{B}_{\mu\nu}{}^{5a}$ as well as the gauge potentials $\hat{B}_{\mu}{}^{5a}$, corresponding to the coset generators M_{5a}, which transform homogeneously according to (19a). Such non-linear actions include,

$$\mathcal{L}_{(1)} = \varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,\hat{B}_{\mu\nu}{}^{ab}\,\hat{B}_{\kappa\lambda}{}^{cd}$$

$$\mathcal{L}_{(2)} = \varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,\hat{B}_{\mu}{}^{5a}\,\hat{B}_{\nu}{}^{5b}\,\hat{B}_{\kappa\lambda}{}^{cd}$$

$$\mathcal{L}_{(3)} = \varepsilon^{\mu\nu\kappa\lambda}\,\varepsilon_{abcd}\,\hat{B}_{\mu}{}^{5a}\,\hat{B}_{\nu}{}^{5b}\,\hat{B}_{\kappa}{}^{5c}\,\hat{B}_{\lambda}{}^{5d}$$

$$\mathcal{L}_{(4)} = \varepsilon^{\mu\nu\kappa\lambda}\,\hat{B}_{\mu\nu}{}^{5a}\,\hat{B}_{\kappa\lambda}{}^{5b}\,\eta_{ab} .$$

All these Lagrangians are invariant under the full non-linear action of $SO(2,3)$. In the unitary gauge the invariance is reduced to the $SO(1,3)$ subgroup with $\hat{B}_{\mu\nu}{}^{ab} \to B_{\mu\nu}{}^{ab}$, $\hat{B}_{\mu\nu}{}^{5a} \to B_{\mu\nu}{}^{5a}$ and $\hat{B}_{\mu}{}^{5a} \to B_{\mu}{}^{5d}$. Notice that in the unitary gauge we may identify the terms in $\mathcal{L}_{(1)}$, $\mathcal{L}_{(2)}$ and $\mathcal{L}_{(3)}$ with those occurring in the model of West and Stelle and they give the Einstein action together with a cosmological term. The action $\mathcal{L}_{(4)}$ becomes $\varepsilon^{\mu\nu\kappa\lambda}\,B_{\mu\nu}{}^{5a}\,B_{\kappa\lambda}{}^{5b}\,\eta_{ab}$ where

$B_{\mu\nu}{}^{5a} = D_\mu B_\nu{}^{5a} - D_\nu B_\mu{}^{5a} = m(D_\mu h_\nu{}^a - D_\nu h_\mu{}^a)$. Now with the interpretation that the $h_\mu{}^a$ are the vierbein fields it follows (Appendix C) that $S_{\mu\nu}{}^\kappa = h^\kappa{}_a(D_\mu h_\nu{}^a - D_\nu h_\mu{}^a)$ is the torsion tensor field which doesn't occur in gravitational actions, (rather it occurs in the algebraic field equations for the $B_\mu{}^{ab}$ - Appendix C). We therefore do not expect $\mathcal{L}_{(4)}$ to contribute to a theory of Einstein-Cartan gravity.

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